This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

1, Mechanics, Analytical, 1908

S n D

mpl

Slocum

PBF

Creu

1, Mechanics, Analytical, 1988

S ה D

Slocu

THE PRINCIPLES OF MECHANICS

THE PRINCIPLES OF MECHANICS

FOR STUDENTS OF PHYSICS AND ENGINEERING

BY
HENRY CREW, PH.D.
FAYERWEATHER PROFESSOR OF PHYSICS IN
NORTHWESTERN UNIVERSITY

LONGMANS, GREEN, AND CO.
91 AND 93 FIFTH AVENUE, NEW YORK
LONDON, BOMBAY, AND CALCUTTA
1908

Stanhope Press
F. H. GILSON COMPANY
BOSTON, U.S.A.

PREFACE

THE following pages represent a lecture course which during several years past has been given to second-year students in physics at Northwestern University.

The prerequisites have been a course in general physics and a course, either concurrent or antecedent, in the calculus.

The author's efforts have been: —

(*i*) To lead the student to clear dynamical views in the shortest possible time, without sacrificing him upon the altar of logic, yet pursuing a route which he can afterwards follow with safety.

(*ii*) To build the discussion upon a few simple experiments and upon definitions which convey at once the physical meaning of the quantities defined. Thus, torque is introduced, not as the vector product of force and distance, but as the time-rate of variation of angular momentum. Likewise moment of inertia is presented at the outset as the rotational inertia of a rigid body, and not as the integral of the second moment of the mass.

(*iii*) To follow the example of Föppl in using vector analysis merely to present a clear, simple, and accurate picture of the facts, reserving the Cartesian analysis for purposes of computation.

(*iv*) To confine the treatment to that part of mechanics which is common ground for the physicist and the engineer.

Kimball 2 Dec 1912

(*v*) To reduce the inherent difficulties of the subject to a minimum by treating dynamics in two analogous parts — rotational and translational — such that if either one is given the other may be immediately deduced. This point of view — this parallel treatment — which carries with it great economy of thought is already very old — dating at least from the introduction of generalized coördinates — but seems to have been fully utilized in few of our modern textbooks.

(*vi*) To employ only two systems of units, the absolute C. G. S. and the "British Engineers."

All teachers of dynamics, it may fairly be supposed, hope to convince the student that the entire science of mechanics is practically nothing else than the application of Newton's Three Laws of Motion and the Principle of the Conservation of Energy to certain special systems, subject to certain boundary conditions. Since each of these laws is subject to mathematical expression, the science is quantitative throughout. From Galileo to Hertz, geometry and analysis have shown themselves indispensable to mechanics.

If this be true, the all-important matter for the engineer and the physicist is to see that the subject does *not* degenerate into a mere set of problems in illustration of the integral calculus and to see that equations are used only to quantify certain experimental results and to predict certain facts of nature. Which of these viewpoints shall dominate the classroom depends very little upon the text employed, very largely upon the attitude of the instructor.

In conclusion, the author wishes to express his obligations to Professor Malcolm McNeill of Lake Forest

University; to Mr. Gordon S. Fulcher, Fellow in Physics at Clark University; to Mr. Harold Stiles, Fellow in Physics at Northwestern University, and to his colleague Professor Robert R. Tatnall for many valuable suggestions and corrections.

H. C.

EVANSTON, ILL.,
March 30, 1908.

TABLE OF CONTENTS

CHAPTER I.

KINEMATICS.

CHAPTER II.

KINETICS.

CHAPTER III.

SOME APPLICATIONS OF GENERAL PRINCIPLES TO SPECIAL PROBLEMS.

CHAPTER IV.

FRICTION.

CHAPTER V.

DYNAMICS OF ELASTIC BODIES.

CHAPTER VI.

FLUID MOTION.

THE
PRINCIPLES OF MECHANICS.

CHAPTER I.

INTRODUCTORY.

1. In general, the pursuit of any particular study presumes a certain amount of previous training. In the case of Dynamics, it is necessary to assume that the student is more or less familiar with the language of mathematical analysis; and in particular that he has at least a slight acquaintance with Analytical Geometry and with the elements of the Infinitesimal Calculus.

In order to take up simpler matters first, and to make sure that reader and writer are each using the same vocabulary, a brief résumé of the geometry of motion — generally called Kinematics — is here given. The necessity for introducing Kinematics at this point lies also in the fact that Dynamics is an attempt to draw a picture of the physical universe in the briefest and most accurate terms; and many of the best terms for this purpose are those of Kinematics.

The masses, forces, and energies which are studied under the head of Dynamics proper, do not here come up for consideration; for Kinematics deals only with the motions of mathematical points and of geometrical figures. Kinematics is not, therefore, a branch of physical science, but belongs rather to mathematics. It is a science which was described by Lagrange as the geometry of four dimensions, because it deals with four, and only four, independent variables, namely, time and the three space coördinates which determine the position of a point. The name, Kinematics, is due to Ampere.

MATERIAL PARTICLES.

2. A *body* is a limited portion of matter; but, when we have to deal with a body which is so small that its dimensions may be neglected in comparison with other distances involved, we call this body a *material particle.* Thus the radius of the earth is so small in comparison with the radius of its orbit about the sun, that the astronomer ordinarily treats the earth as a particle; while for the geologist the earth is a large and important body. The shot of a 13-inch rifle is treated as a particle by one who is computing its range; but for the man who is shaping the shot on the lathe, or for the sailor who is loading it upon the ammunition-hoist, it is a body of very considerable dimensions.

In what immediately follows we shall have frequent occasion to speak of particles and of bodies; but during the entire treatment of Kinematics we shall consider them both as massless; in other words, we shall for the present employ the word "particle" to denote a mathematical point, assigning to it only the property of position; in like manner the word "body" will be used to denote a geometrical figure. Later, when we reach the subject of Dynamics proper, we shall see that the mass of a body is an all-important factor.

POSITION OF A PARTICLE.

3. Since motion is merely change of position, it becomes necessary, at the very outset, to define position. Everyone understands that position is a relative term, and that the position of a particle can be described only with reference to some point whose position we assume to be known. This reference point being fixed, the position of any particle relative to it is usually defined by one of the two following methods:

(i) *Rectangular Coördinates.* The method of rectangular coördinates was first introduced by Descartes in 1637. By it, one defines the position of any particle, say *P*, Fig. 1, by giving the perpendicular distances of *P* from three mutually perpendicular planes which intersect at the point of reference, *O*, whose position is known or assumed.

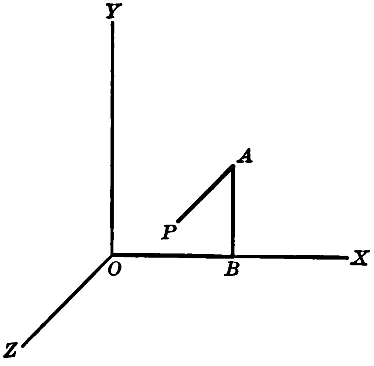


Fig. 1.

These three mutually rectangular planes, taken two at a time, intersect in the lines *OX*, *OY*, and *OZ*. These three lines are known as the *axes* of *X*, *Y*, and *Z* respectively.

Thus, in Fig. 1, *PA* is the perpendicular distance of the particle *P* from the plane which includes the axes *OX* and *OY*, and which is, therefore, called the *XY-plane.* In like manner, *AB* is the distance of the particle *P* from the *XZ*-plane, and *BO* is the distance of *P* from the *YZ*-plane.

Distances measured parallel to the axes of X, Y, and Z, are generally denoted by the letters x, y, and z, respectively. Thus, in Fig. 1, the distance OB is denoted by x; BA by y; and AP by z. These distances are called the *rectangular coördinates* of the point P.

Values of x measured to the right of the YZ-plane are called positive; those measured to the left are indicated by a negative sign. In the same manner, values of y measured upwards are called positive; those measured downwards, negative. Values of z are positive when measured outwards, i.e., towards the observer; they are negative when measured away from the observer.

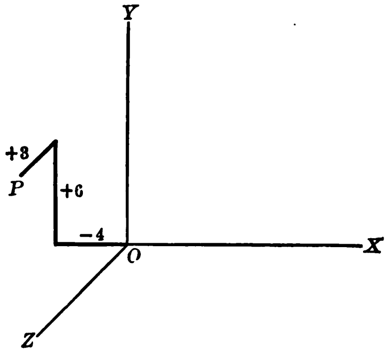


Fig. 2.

The three coördinates of a point thus given define the position of the point without ambiguity. If, for instance, it were required to locate the point $(-4, +6, +3)$, one might proceed as indicated in Fig. 2.

(ii) *Polar Coördinates.* In order to define the position of a particle P with respect to a given point O, it is, in many cases, most convenient to describe a sphere about O as center, Fig. 3, choosing the radius, r, such that the particle P will lie on the surface of the sphere. The radius of the sphere is then the distance PO. If in addition to this we give the latitude and longitude of the particle P on the surface of this sphere, its position is completely determined. The longitude of P is described by assuming a point N on the surface of the sphere which we may call its *pole*, and by assuming a fixed plane of reference which passes through the pole and through the center of the sphere. Let *NOE*, Fig. 3, be this plane. The angle which the plane through *N*, *O*, and the particle *P* makes with this reference plane is called the *longitude* of the particle. We shall denote it by ϕ.

But the radius, r, and the longitude, ϕ, suffice merely to locate the particle, *P*, on a definite meridian. Now if, in addition to these, the angle *NOP* is given, the position of *P* is completely determined. This angle *NOP* is the complement of the latitude of *P*, and is, therefore, called the *colatitude* (or polar distance) of *P*. It is generally denoted by θ. These three data, radius, longitude, and colatitude, r, ϕ, and θ, are called the *polar* (or *spherical*) *coördinates* of a point.

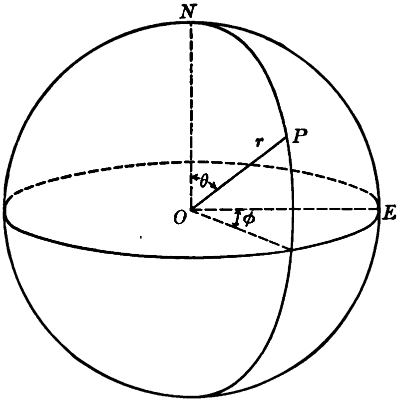


Fig. 3.

It will be observed that r is here merely the distance from the center to the surface of the sphere, and has the same value in whatever direction it is measured. The *direction* of the point *P* from *O* is determined by the values of ϕ and θ.

It is interesting to observe also that while ϕ varies between the limits 0° and 360°, θ can vary only from 0° to 180°.

VECTORS.

4. Any straight line, such as $\overline{OP}$ in Fig. 4, which has a definite length and is drawn in a definite direction and in a definite sense, is called a *vector*. The vector, $\overline{OP}$, may, therefore, be looked upon as an operator which carries the particle from O to P. The end O is called the *initial point*, the end P the *terminal point*, of the vector.

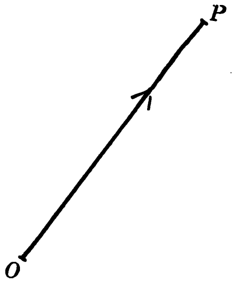


Fig. 4.

Let us now denote the definitely directed line $\overline{OP}$ by $\bar{r}$; then $\bar{r}$ is a position vector which completely defines the position of the point P with reference to the point O.

Whenever and however P moves with respect to O, $\bar{r}$ will change value because the symbol $\bar{r}$ carries with it both distance and direction. The *sense* in which a vector is drawn is usually indicated by an arrow; this simply tells which of the two ends is the initial point and which the terminal point.

As we shall see in the two following sections, a vector may be numerically defined by the use of either rectangular or polar coördinates. And still later we shall find that there are many other physical quantities besides position that can be clearly represented by vectors such as OP in Fig. 4. Velocity, momentum, acceleration, force, etc., are quantities of this type, having both direction and amount; they are therefore called *vector quantities.* Indeed, *any* quantity which is related to a definite direction in space is called a vector quantity. It is important to observe that this value of a vector is quite independent of the position of its initial point; so long as the *direction*, *sense*, and *amount* of a vector remain unchanged, its value remains unchanged.

DISTINCTION BETWEEN SCALARS AND VECTORS.

5. It is important, however, to note that not all quantities possess direction. Thus any one of the following quantities is devoid of direction: the volume of a sphere; a pure number, such as 5; a temperature; a length as indicating mere size, such as a 2-inch screw; a quantity of energy, a mass of iron. Quantities of this kind are capable of being completely represented by a mere distance on a scale, and are, therefore, called scalar quantities, or, more simply, *scalars.*

In the case of quantities which have direction, the matter is entirely different; here the single quantity, say $\bar{r}$, requires three specifications to completely define it, namely, a distance, a direction, and a sense. Thus the position of one boat with respect to another, or the velocity of the wind, will each require three data for their description. The same is true of an electrical, magnetic, or gravitational field of force; it is true, indeed, of all forces and velocities. These illustrations will serve, perhaps, to indicate the general character of vector quantities.

It is well to remember also that vectors quantities are more general than scalars. All vectors contain a scalar factor. Any vector may be considered as a scalar multiplied by a certain directed quantity whose numerical value is unity. Thus the velocity of a train running N. E. at the rate of 17 miles an hour is made up of the scalar factor "17" and the vector factor "1 mile an hour N. E."

TO DESCRIBE THE DIRECTION OF A VECTOR.

6. For this purpose one of the most frequently used methods is to give the three *direction-cosines* of the vector, that is, the cosines of the three angles which it makes with the axes of X, Y, and Z respectively.

Thus, in Fig. 5, the values of cos α, cos β, and cos γ, constitute three numbers which definitely define the direction of the vector $\overline{OP}$.

From this it might appear that four specifications are necessary to define the vector $\overline{OP}$, namely, its length r and its three direction-cosines. But it must be remembered that these last three quantities are not independent of each other, since the following relation

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \qquad \text{Eq. 1}$$

always exists between them. This equation is merely the trigonometrical method of saying that the square on the diagonal of any rectangular parallelopipedon is equal to the sum of the squares on its three edges. When, therefore, any two of these cosines are given, the third is thereby determined. Accordingly we have the general proposition that three, and only three, independent data are needed to locate a point in space.

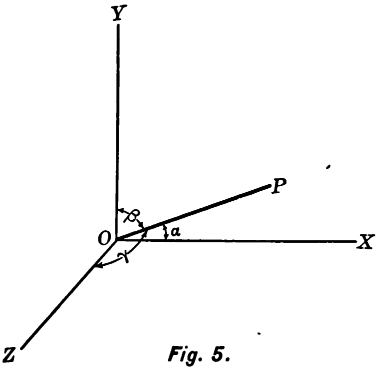


Fig. 5.

If, instead of the polar coördinates, the rectangular coördinates of the terminal point are given, then

$$r^2 = x^2 + y^2 + z^2 \qquad \text{Eq. 2.}$$

which determines the length of the vector, and

$$\left.\begin{aligned}\cos\alpha &= \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2+z^2}},\\ \cos\beta &= \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2+z^2}},\\ \cos\gamma &= \frac{z}{r} = \frac{z}{\sqrt{x^2+y^2+z^2}},\end{aligned}\right\} \qquad \text{Eq. 3.}$$

which determine its direction cosines.

Or, by projecting the rectangular coördinates upon the radius vector, r, we obtain the following mixed value for its length, namely,

$$r = x\cos\alpha + y\cos\beta + z\cos\gamma. \qquad \text{Eq. 4.}$$

Transformation of Coördinates.

7. When the position of the particle is defined by polar coördinates, we may, by the following simple transformations (see Fig. 3), pass to rectangular coördinates.

$$\left.\begin{aligned}x &= r\sin\theta\cos\phi,\\ y &= r\cos\theta,\\ z &= r\sin\theta\sin\phi.\end{aligned}\right\} \qquad \text{Eq. 5.}$$

By Eq. 5 we evaluate x, y, and z, and then use Eq. 3 to determine the direction cosines; or we may consider the direction as already determined by the values of ϕ and θ.

Angle between Two Vectors.

8. When the directions of two vectors, $\bar{r}_1$ and $\bar{r}_2$, are defined by the angles $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ respectively, the angle between the two vectors, θ, is given by the following equation:

$$\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2. \qquad \text{Eq. 6.}$$

This result may be obtained almost immediately by first projecting the vector $\bar{r}_1$ upon $\bar{r}_2$; then projecting the components of $\bar{r}_1$ upon $\bar{r}_2$, and then equating these two projections. In this way we obtain

$$r_1 \cos \theta = x_1 \cos \alpha_2 + y_1 \cos \beta_2 + z_1 \cos \gamma_2.$$

Substituting now the values of the components, namely,

$$\left.\begin{aligned} x_1 &= r_1 \cos \alpha_1, \\ y_1 &= r_1 \cos \beta_1, \\ z_1 &= r_1 \cos \gamma_1, \end{aligned}\right\}$$

we have the desired theorem.

POSITION OF A BODY.

9. Having thus far considered only the position of a particle, we now take up the position of a *body*, which has already (§ 2) been defined as "a limited portion of matter."

The changes which a body may undergo naturally fall into two classes: (1) changes in which the body is altered neither in shape nor in size; and (2) changes in which the body *is* altered either in shape or in size. In changes of the first type we may consider the body as a perfectly rigid geometrical figure whose motion in general will depend upon the amount, but not upon the kind, of matter which it contains. In changes of the second type, however, where the body is transformed in shape or where its bulk is altered, we shall find the changes depending very largely upon the chemical composition and upon the physical state of the body. Changes of this latter kind will be studied later under the heads of Elasticity and Hydrodynamics.

For the present we shall deal only with such changes as might occur in a perfectly rigid solid, that is, in a body whose size and shape are each constant; and we shall here consider the body merely as a system of rigidly connected particles, that is, a geometrical figure.

Such a figure can be said to be given only when the position of each particle in it is known with reference to any other particle in it. This being so, the position of the body is completely determined when we know *the position of any three points not lying in the same straight line.*

CHANGE OF POSITION: Linear Displacement.

10. Having now considered the position of a particle and the position of a rigid body, the next natural inquiry is concerning *change* of position.

(i) *Case of Particle.* Let us assume a particle at the point A, whose coördinates are (x_1, y_1, z_1), Fig. 6. Its position is then defined by the following four equations, of which only three are independent.

$$p = \sqrt{x_1^2 + y_1^2 + z_1^2},$$

$$l_1 = \frac{x_1}{p},$$

$$m_1 = \frac{y_1}{p},$$

$$n_1 = \frac{z_1}{p},$$

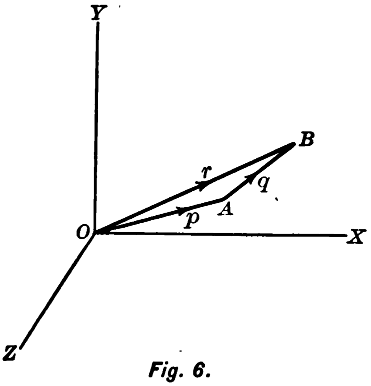


Fig. 6.

where l_1, m_1, n_1, denote the direction cosines of $\bar{p}$. When this particle is moved to another position, say to the point B, whose coördinates are (x_2, y_2, z_2), its *change* of position is measured by the straight line q connecting these two points A and B. From this it is evident that change of position involves direction as well as distance, and is, therefore, a vector quantity.

It is also evident that this second vector is completely defined by the following equations:

$$q = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},$$

$$l_2 = \frac{x_2 - x_1}{q},$$

$$m_2 = \frac{y_2 - y_1}{q},$$

$$n_2 = \frac{z_2 - z_1}{q},$$

where q is the distance between the points A and B, and (l_2, m_2, n_2) the direction cosines of $\bar{q}$.

ADDITION AND SUBTRACTION OF VECTORS.

11. Referring still to Fig. 6, it is also clear that the new position, B, of the particle, which has been reached by the use of two vectors $\bar{p}$ and $\bar{q}$, may be described by another vector, namely, the straight line OB drawn directly from the origin O to the new position, x_2, y_2, z_2. In the language which we have adopted, this vector is completely specified by the following four equations:

$$r = \sqrt{x_2^2 + y_2^2 + z_2^2};$$

$$l_r = \frac{x_2}{r}; \quad m_r = \frac{y_2}{r}; \quad n_r = \frac{z_2}{r}.$$

The position, x_2, y_2, z_2, has thus been described by two different, but strictly equivalent, methods. By one method the particle at B is located as the terminal point of a vector $\bar{r}$; by the other method it is the terminal point of two vectors, $\bar{p}$ and $\bar{q}$, placed end to end (the terminal point of one vector coinciding with the initial point of the following) and with their arrows in the same sense. And since $\bar{r}$ locates the point B at the same distance and in the same direction from the origin, $\bar{r}$ is said to be the *resultant* or *vector-sum* of $\bar{p}$ and $\bar{q}$.

In the algebra of vectors this is expressed by writing

$$\bar{p} + \bar{q} = \bar{r}. \qquad \text{Eq. 7.}$$

Beyond what is contained in this equation, there is little to be said upon the subject of vector addition.

12. The resultant of any number of vectors is obtained in precisely the same manner by placing these vectors end to end, in any order, *but all in the same sense,* and then joining the initial point of the first vector to the terminal point of the last vector. Most students are already familiar with this process under the name of the "parallelogram of velocities" or the "polygon of forces." In ordinary algebra, which deals only with scalars, we should have, referring still to Fig. 6,

$$p + q > r.$$

But this inequation refers merely to lengths which are scalars, and not at all to displacements, which are vectors.

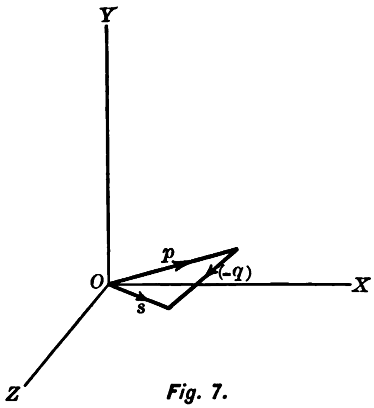


Fig. 7.

13. The addition of vectors once mastered, the subtraction of vectors becomes very simple. We have only to recall that change of sign in a vector means change of sense, or, what amounts to the same thing, a change of 180° in direction. Accordingly the difference of two vectors, say $\bar{p}$ and $\bar{q}$, is merely the sum of ($\bar{p}$) and ($-\bar{q}$). Thus in Fig. 7 let us give to $\bar{p}$ and $\bar{q}$ the same meaning which they had in Fig. 6; then $\bar{s}$ will be the resultant of $\bar{p}$ and $-\bar{q}$, which is expressed by the following equation:

$$\bar{p} - \bar{q} = \bar{s}. \qquad \text{Eq. 8.}$$

Those vectors which are added together to produce any resultant are called the *components* of that resultant. Thus in Fig. 7, $+\bar{p}$ and $-\bar{q}$ are the components of $\bar{s}$. In Fig. 6, $+\bar{p}$ and $+\bar{q}$ are the components of $\bar{r}$. The multiplication of vectors will be taken up a few pages later, just before we shall need to use it. See Digression on Vector Algebra, § 31.

PROBLEMS.

1. What angle must two components, equal in length, make with one another in order that their resultant may be equal in length to each of the components?

2. Interpret the equation

$$\bar{r} - \bar{p} - \bar{q} = 0.$$

3. The adjacent sides of a parallelogram are $\bar{a}$ and $\bar{b}$. Show that its diagonals are $\bar{a} + \bar{b}$ and $\bar{a} - \bar{b}$.

4. The rectangular coördinates of three points lying in a plane are (3, 2), (4, 4), and (1, 2). Find as vectors the three sides of the triangle determined by these three points.

5. A railway train travels from a town A, 50 miles in a northeast direction and reaches a town B; while a second train leaving the town A, runs 30 miles due south and reaches a town C. Find the vector joining the town B with the town C.

6. Draw two vectors such that the scalar part of their resultant is equal to the scalar part of each of the components.

7. How must two vectors be drawn so that the scalar factor of their sum is equal to the scalar factor of their difference?

8. Show by means of vectors that the diagonals of a parallelogram mutually bisect each other.

14. (ii) *Case of a Rigid Body.* We shall first consider two of the simplest possible changes of position, and then proceed to show that any motion whatever may be reduced to a combination of these two simple motions.

Pure Translation: Linear Displacement.

15. When a body moves in such a way that each particle in it traverses a path equal and parallel to that traversed by every other particle, the motion is said to be a *pure translation.* The cross-head of a piston rod in the ordinary steam engine very closely approximates this motion. Perhaps the motion of the carriage on a dividing engine such as those employed for ruling diffraction gratings is the nearest approximation known to pure translation. Evidently translation may be considered a special case of rotation, namely, a rotation about an infinitely distant axis, Since in a case of pure translation of a rigid body every particle traverses a similar path, it is evident that the displacement of the body is to be described and computed in exactly the same manner as the displacement of a particle treated in §§ 10–13. It is very important to observe that in pure translation the path is not necessarily straight. It may, indeed, be circular; but if circular for one particle, then all other particles traverse a circle of the same size.

Analogue. Pure Rotation: Angular Displacement.

16. When any point in a body, or rigidly connected with a body, remains fixed while the body moves, the motion is said to be one of *pure rotation.* In such a motion there is always one line of particles (or points) which at any instant remains fixed. This line is called the *instantaneous axis of rotation.* But, if there are two points in a body which are permanently fixed, then the line defined by these two points is also permanently fixed, and is called the *fixed axis* of rotation. A wheel turning between two pivots, as in Fig. 8, typifies this change of position. It is evident that in any pure rotation each particle in the body (which we here assume to be rigid) passes from any given position to any other position by moving along the arc of a circle which has its center on the axis of rotation. Of all the changes of position which occur in machines, this is the one most frequently met. Many examples such as the following will occur to everyone: the spindle of a lathe, the armature of a dynamo, the balance wheel of a watch, a grindstone, especially the stone used for beveling plate glass, such as that shown in Fig. 8.

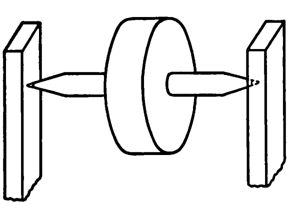


Fig. 8.

Analogue. Change of Position: Angular Displacement.

17. Imagine any rigid body having one point O fixed. Take this point O as origin. Suppose the body to turn through any angle θ about an axis OP. This is an angular displacement of the kind which we have just called a pure rotation. It requires for its description three specifications:

namely, the direction of the axis of rotation; the amount of rotation, i.e., the angle through which the body has turned; and the sense of rotation. In short, this angular displacement requires a vector to represent it. And in this respect it is strictly analogous to linear displacement.

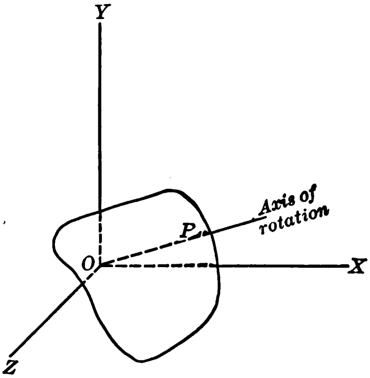


Fig. 8 bis.

Thus an angular displacement $\bar{\theta}$ of a rigid body — including a particle — about an axis *OP* (Fig. 8 *bis*) is completely specified by laying off a vector, say $\bar{\theta}$, along *OP* whose scalar part shall be numerically equal to the number of radians in $\bar{\theta}$, and whose sense shall be such that when one looks along the vector in the sense indicated by the arrow the rotation shall appear to be clockwise; in other words, between the vector and the angular displacement there exists the right-handed screw relation.

Composition and Resolution of Angular Displacements.

18. One might at first glance think that since angular displacements are represented by vectors they would be compounded and resolved exactly as linear displacements. And indeed this is true so long as the angular displacements are small, or when they are simultaneous and of any size. But a simple experiment such as the following will illustrate the difference when one is dealing with *finite* displacements.

Hold a stiff hat in your hand in any fixed position, and imagine a set of rectangular axes drawn in it; and then compare the effects of the two following operations.

1. Rotate the hat in a clockwise sense through 90°, from its initial position, about the axis of X; then follow this by an equal rotation (90°), in the same sense, about the axis of Y. Note the final position of the hat.

2. Bring the hat back to its initial position and repeat these same two operations, only in reverse order. It will be observed that the final position of the hat is quite different from that in the first case. In other words, if the vectors are finite, the resultant depends very much upon the order in which the components are added. As may be easily verified with the hat, this is not the case when the angular displacements are very small.

If both rotations occur at the same time, we may consider the position of the hat at any instant as the result of three infinitesimal rotations about the axes of X, Y, Z during the infinitesimal interval of time just preceding the present moment. Accordingly finite simultaneous angular displacements are compounded just as linear displacements, and hence call for no further discussion here. Try the experiment with your open watch or Derby hat, or some other easily oriented object. The illustrative problems following will make the matter perfectly clear.

PROBLEMS.

1. Draw a vector which shall represent a clockwise angular displacement in a horizontal plane viewed from above.

2. What is the difference between two vectors, one of which represents the displacement of the minute hand, the other of the hour hand of a watch during the interval between two and four o'clock?

3. Which has the larger scalar part, a vector which represents a linear displacement of six units of length, or one which represents an angular displacement of one complete rotation?

4. If the foot be taken as the unit of length, how long must a vector be drawn to represent an angular displacement of 5°?

5. A particle moves in a circle. Find its linear displacement when the angular displacement of the radius joining it to the center of the circle amounts to 60°, 90°, 180°, 240°.

6. A body receives an angular displacement of 3″ about the axis of X, and a simultaneous angular displacement of 4″ about the axis of V. What vector will represent the resultant displacement?

CONSTRAINED MOTIONS.

19. Hitherto we have been considering the motion of a body which is perfectly free to move in any manner whatsoever. But, as a matter of fact, nearly all the motions which are met with in actual engineering practice, or in the physical laboratory, are motions in which the body is compelled (by forces applied) to move along certain surfaces or to rotate about definite fixed axes. Such motions are said to be *constrained.* Their discussion forms a large chapter in Applied Mechanics. That constrained motion which is, by all odds, the most important, and the one which includes perhaps the great majority of all the motions encountered in actual mechanisms, is that in which a rigid body is compelled to move so that any one section of it continues to move always in its own plane. One of the simplest illustrations of such a motion is that of a coin or plate sliding over a table. Such a motion is said to be uniplanar.

20. (i) *Plane Motion.* The most important property of this motion the student will be able to prove for himself. It is as follows: *Any displacement of a rigid body parallel to* *one plane may be produced by rotation about a definite axis perpendicular to that plane.*

It is evident from a consideration of Fig. 9 that any straight line BA can be moved into any other position $B'A'$ in its own plane by a pure rotation about the point C, where C is determined as follows: Join the corresponding points A and A' by a dotted line; do the same for the points B and B'. Bisect these dotted lines and erect perpendiculars at the points of bisection. The intersection of these perpendiculars is C, the required center of rotation. For since CB is equal to CB' and CA to CA', it is clear that the two triangles ABC and $A'B'C$ are equal; and since they have a common vertex at C, they can be made to coincide by a pure rotation.

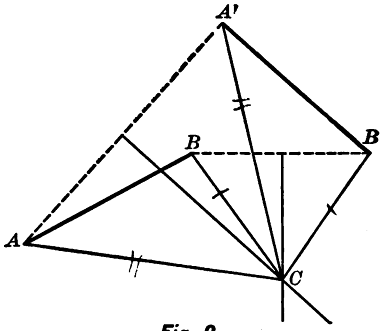


Fig. 9.

The student will observe that the theorem just enunciated does not undertake to describe how the body BA actually moved into the position $B'A'$. It merely states how this change of position *might* have been accomplished, namely, by rotation about the center C, which for this reason is sometimes called the *virtual center.*

PROBLEM.

In the case of an ordinary stationary steam engine, what motion can you think of which is *not* uniplanar?

Most General Displacement of a Rigid Body.

Chasles' Theorem.

21. (ii) *Screw Motion.* A little consideration will show that *any rigid body whatever may be moved from any one position to any other position by a single translation and a single rotation,* i.e., by translation along and rotation about a single axis. The demonstration of this theorem is as follows:

In Fig. 10, let ABC be one position of three points of any body whatever, and $A'B'C'$ another position of the same three points. Imagine a sphere described about the point A as center. (1) The plane determined by the three points ABC will cut this sphere in a great circle. (2) Draw another plane $AB''C''$ passing through A, and parallel to the plane $A'B'C'$, and make AB'' parallel to $A'B'$,

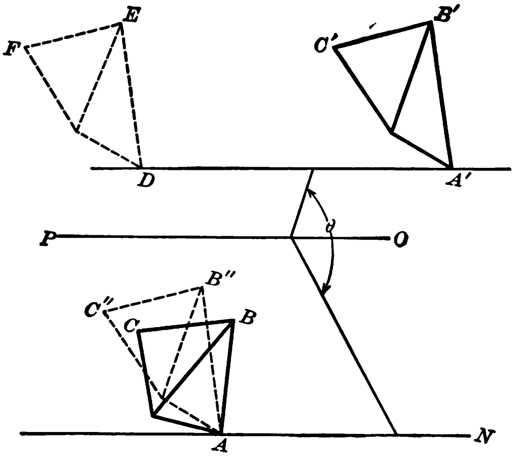


Fig. 10.

also AC'' parallel to $A'C'$; this plane $AB''C''$ will also cut the surface of the sphere in a great circle. The angle BAC will determine a definite arc on this sphere; so also will the angle $B''AC''$ determine an arc of equal length.

It is now an easy matter, by the method just described under plane motion, to rotate one of these arcs into coincidence with the other.

Let us denote by NA the axis through A about which the body must be rotated in order to produce this coincidence; and by θ the angle through which the body must be thus rotated. Then it is clear that, if we rotate ABC through an angle θ about any axis *parallel* to NA, the planes ABC and $A'B'C'$ will still be brought into parallelism. Through A' draw the line DA' parallel to NA. We will now choose for an axis of rotation, a line parallel to NA, such that when the figure ABC is rotated through angle θ, the point A will coincide with the point D. Let the axis thus determined be indicated by the line OP. In general, OP will not lie in the same plane with NA and DA'. We can now imagine the body to be rotated about the axis OP, and, at the same time, to be translated along *this same axis* through a distance DA'. We have thus reduced the most general displacement of a body to a pure translation and a pure rotation *about the same axis.* When these two motions occur simultaneously, the resulting motion is called a *twist* about a screw, or simply a *screw motion.* The pure translation and pure rotation previously discussed are evidently special cases of twist.

We have then the general theorem that any displacement whatever of a rigid body in space may be accomplished by a simple screw motion.

Incidentally it is very interesting to note that in any rigid body there is always one set of planes whose orientation is not changed by any *single* displacement; that is, the position of one set of planes after displacement is parallel to its position before displacement. Referring to Fig. 10, it is clear that this set of planes is the one perpendicular to the axis NA; for their orientation is not changed by the rotation θ, nor by the translation DA'.

The upshot of the whole matter is that a rigid body *may* be moved from any one position to any other position by a screw which has the proper pitch, and has its axis in the proper direction. The two motions (uniplanar and screw) which have just been discussed are the most useful and most frequently encountered of all constrained motions.

DEGREES OF FREEDOM.

(1) Case of a Particle.

22. It is interesting to observe that when a particle is perfectly free to move in any direction whatever, it can move in three, and only three, directions, such that its motion along any one of these directions has no component along either of the other two. And it is evident that these three directions are mutually perpendicular; for if they were inclined at any angle other than 90°, say θ, a displacement r along one axis would have a component $r \cos \theta$ along the other. These three directions are represented, therefore, by any three mutually perpendicular axes. In view of these facts, a particle is said to have three *degrees of freedom.* In other words, a particle has three possibilities in the way of independent translational motion. It cannot rotate, because, by definition, it has no appreciable dimensions.

(I) Some Important Special Cases.

86. *Case* 1. *A uniform fine straight wire.* Find its moment of inertia about an axis perpendicular to its length, and passing through its middle point.

Let L = total length of wire.
λ = mass of unit length = linear density.
dr = element of length.
Then $dM = \lambda\, dr$.
λL = total mass = M;

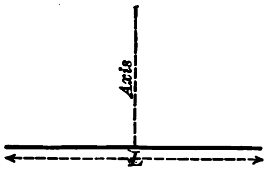


Fig. 34.

and one obtains, Eq. 80,

$$I = \int_0^M r^2\,dM = 2\lambda \int_0^{\frac{L}{2}} r^2 dr = 2\lambda \left|_0^{\frac{L}{2}} \frac{r^3}{3} = \frac{\lambda L^3}{12} = \frac{ML^2}{12}.\right. \quad \text{Eq. 81.}$$

The student will find it an interesting exercise to prove that, when the axis of rotation is shifted parallel to itself, so as to pass through one end of the wire, the expression for the rotational inertia becomes

$$I = \frac{ML^2}{3}. \qquad \text{Eq. 82.}$$

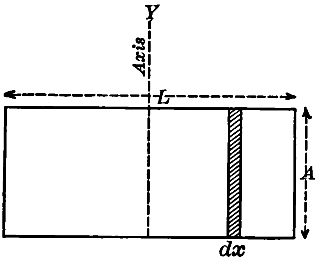


Fig. 35.

87. *Case* 2. *A uniform rectangular plane lamina.* Find the moment of inertia about a central axis lying in its plane and perpendicular to one side, say L.

Let us here use rectangular coördinates, such that the axis of Y coincides with the axis of rotation. Then we may choose for our element of mass a narrow strip parallel to the axis of Y, and having a width dx. Accordingly,

$$dM = \sigma y \,.\, dx = \sigma A \,.\, dx,$$

where σ = the surface density of the lamina, — i.e., the mass of unit area of the rectangle, — and A is the length of side parallel to the Y-axis.

Since $x = r$ = distance of dM from the axis, our integral becomes

$$I = \int_0^M r^2\,dM = 2\int_0^{\frac{L}{2}} x^2 \,.\, \sigma \,.\, A \,.\, dx = 2\,\sigma A \left| {}_0^{\frac{L}{2}} \frac{x^3}{3} = \frac{\sigma A L^3}{12}. \right.$$

But since $M = \sigma A L,$

$$I = \frac{ML^2}{12}, \qquad \text{Eq. 83.}$$

a result which might have been anticipated from the fact that a plane lamina of this kind may be considered as built up of exactly similar elements of length L and width dy, each of which would fall under Case 1.

It is worth noticing that the rotational inertia of such a lamina is a function of its depth A, only in so far as M depends upon A.

88. *Case* 3. *A circular ring of uniform wire.* Find the moment of inertia about a central axis perpendicular to the plane of the circle.

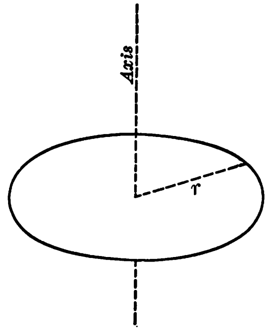


Fig. 36.

Here r, the radius of the ring, is a constant for each element of mass. Accordingly, our integral becomes

$$I = \int_0^M r^2\,dM = r^2 \int_0^M dM = Mr^2. \qquad \text{Eq. 84.}$$

the experiments of Lavoisier and others, that matter has a continuous existence, and, in moving from one position to another, must therefore occupy every intermediate position along the route. In mathematics one often meets a point traveling along a curve that has gaps — so-called discontinuities — in it. But in dynamics no such motion is ever encountered.

The *path* of a particle is defined as the continuous locus of all the successive positions which it occupies.

Description of Motion of Point. Definition of Speed.

26. The rate at which a particle moves along its path is called the *speed* of the particle.

Imagine a man traveling along a narrow path which has no branches. The motion of the man will be described in a definite way only when we give (explicitly or implicitly) the complete location of the path (its geometry) and the time at which he reaches each particular point in the path. Let us denote by s the distance from the beginning of the path to the point where the man is at any particular instant of time t. Then the data necessary to describe the man's motion are (1) the geometry of the path and (2) the distance s as a function of the time t. So, in the motion of a particle, all we need to know is the path, and the distance which the particle has traveled along the path at each particular instant. This latter fact would be naturally expressed by an algebraic equation as follows:

$$s = f(t). \qquad \text{Eq. 9.}$$

The rate at which s is changing as t varies is what we have above called the speed of the particle. Evidently this is obtained by differentiating s with respect to t. Accordingly,

$$\text{Speed at instant } t = \frac{ds}{dt} = f'(t). \qquad \text{Eq. 10.}$$

Now s is a scalar quantity, being a mere distance without regard to direction; and so also is t. The differential coefficient, $\frac{ds}{dt}$, to which we have given the name speed, is, therefore, also a scalar quantity, being itself the limit of a ratio of two scalar quantities.

A Second Mode of Description.

27. Instead of giving the geometry of the path and the distance of the man from the beginning of the path, we might have given the geometry of the path and the *speed* of the man at each instant.

$$v = \phi(t).$$

Using these data, one obtains the length of the path from the speed by integration, that is, by summing up the elements of the path, $v\,dt$, as follows:

$$s = \int_0^t \phi(t)\,dt.$$

Definition of Velocity.

28. It is evident, of course, that when a particle moves, it always moves, at any particular instant, in some one direction, *and* in some one sense. To completely describe the motion at any instant, it is necessary, therefore, to know not only the speed, but also the direction and the sense of the motion at that instant. This can be most conveniently done by use of a vector which shall be drawn in the same direction and sense as that of the motion, and which shall have a length equal to the speed of the motion. Such a vector is called the *velocity* of the particle at the particular instant under consideration.

Distinction between Speed and Velocity.

29. Speed represents merely the rapidity of motion, while velocity represents both rapidity and direction. Speed is defined by a single number, while velocity demands two specifications, namely, amount and direction. Imagine two trains traveling along the same piece of double track, the one going north 30 miles an hour, the other going south 30 miles an hour. Both trains have the same speed; but their velocities are as different as they can be — that is, they are directly opposite in sign.

To denote speed, we shall employ the symbol v; but to denote velocity, we shall use the vector symbol $\bar{v}$.

Resolutions and Composition of Velocities.

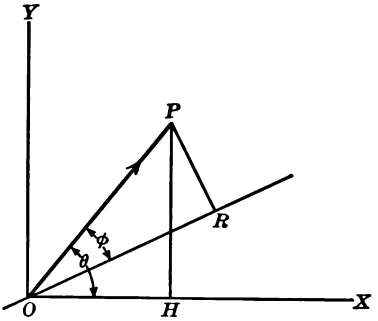


Fig. 11.

30. Since velocity is a vector quantity, it is evident that we may at once apply to it the general rules for vector addition and subtraction.

Thus, having given a particle whose speed is v and whose direction is defined by θ, its velocity may be represented by the vector $\overline{OP}$, in Fig. 11. Let us suppose the motion to take place in the XY- plane. Then if we desire the velocity of the particle in a direction parallel to the axis of X, it is only necessary to project OP on the axis of OX. We thus obtain OH as the required velocity. In like manner HP will be the velocity parallel to the axis of Y. If the reference axes are fixed, then

$$\left.\begin{aligned} OP\cos\theta &= OH = v_x = \frac{dx}{dt}, \\ OP\sin\theta &= HP = v_y = \frac{dy}{dt}. \end{aligned}\right\} \qquad \text{Eq. 11.}$$

Also $$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2. \qquad \text{Eq. 12.}$$

To compute the velocity along any direction OR, making an angle ϕ with OP, we have

$$OP\cos\phi = OR = v_x\cos(\theta-\phi) + v_y\sin(\theta-\phi). \qquad \text{Eq. 13.}$$

If the speed and direction of motion are each constant, we have

$$\bar{v} = \frac{d\bar{s}}{dt} = \frac{\bar{s}}{t} = \text{constant} = \textit{actual velocity}. \qquad \text{Eq. 14.}$$

If neither speed nor direction is constant, then we have

$$\frac{s}{t} = \textit{average speed}. \qquad \text{Eq. 15.}$$

We shall have very frequent occasion to use these expressions in computing the values of s, v, and t; also in defining the unit of speed.

There is a slightly different and very instructive point of view from which this entire subject of velocity may be regarded; but no discussion of this would be clear without some consideration of vector multiplication, a subject to which we now proceed.

PROBLEMS.

1. Having given the velocities of a particle along the axes of X and $Y \left(\frac{dx}{dt} \text{ and } \frac{dy}{dt}\right)$, find the components, of the same velocity when resolved along axes X' and Y', which make a constant angle θ with the former axes. See Fig. 12.

2. A train is running north with a speed of 60 feet per second. A ball is thrown from the train in a direction north 30° east with a speed of 30 feet per second. Find the velocity of the ball with respect to the ground.

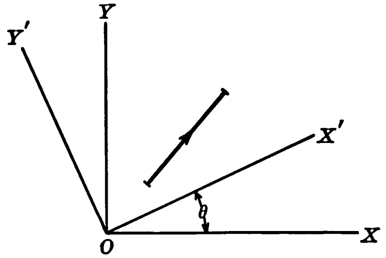


Fig. 12.

3. A boat is rowed directly across a river with a speed which is just twice that of the current. It reaches the opposite bank of the river ½ mile below the starting-point. Find the breadth of the river.

Digression on Vector Algebra.

31. In the following chapters we shall have frequent occasion to add, subtract, multiply, and divide certain vectors. The elementary principles according to which these operations are performed constitute a branch of mathematics which is called *vector analysis,* and which is simply the Algebra of vector quantities. It differs from ordinary algebra only in the three following features:

(1) The Unit Vector.

The quantities which are especially treated in vector analysis are called vectors, and are here denoted by a dash over the symbol. Now any vector is, *in a certain* *sense*, a physical quantity, and is, therefore, like other physical quantities, measured in terms of certain units. Each of these units is a vector whose scalar value is unity, and whose direction is the direction of the vector to be expressed. We shall here indicate unit vectors by i, generally with a subscript which will show its direction.

Thus,

$$\bar{F} = F\bar{i}_f. \qquad \text{Eq. 16.}$$

In Eq. 16, $\bar{F}$ is a vector which may be expressed as the product of two factors, namely, F and $\bar{i}_f$, the former being an abstract number which has the same numerical value as $\bar{F}$, and the latter a unit vector having the same direction as $\bar{F}$. This purely numerical quantity, F, by which the unit vector must be multiplied to produce the complete vector $\bar{F}$, is sometimes called the *tensor* of $\bar{F}$.

Special Type of Unit Vector.

32. When a vector is resolved into three mutually perpendicular components, say $\bar{x}$, $\bar{y}$, and $\bar{z}$, these three quantities are expressed each in terms of a unit vector along its particular axis; and unit vectors along the axes of x, y, and z are practically always denoted by $\bar{i}$, $\bar{j}$, and $\bar{k}$, respectively.

Hence the three components above mentioned would be written as follows:

$$\bar{x} = x\bar{i}, \quad \bar{y} = y\bar{j}, \quad \text{and} \quad \bar{z} = z\bar{k}. \qquad \text{Eq. 17.}$$

And, according to the principle of vector addition which we have already studied, the resultant vector, $\bar{R}$, is

$$\bar{R} = x\bar{i} + y\bar{j} + z\bar{k}.$$

In what follows, we shall assume that the student is familiar with the rules for the addition and subtraction of vectors as set forth in the earlier pages of this treatment and as expressed in the following equation,

$$\overline{F} = \bar{a} + \bar{b} + \bar{c} + \bar{d}, \qquad \text{Eq. 18.}$$

whose meaning will be clear from Fig. 13. It is evident that quantities may be transferred from the right to the left side of this equation, and *vice versa*, by merely changing signs, as in ordinary algebra.

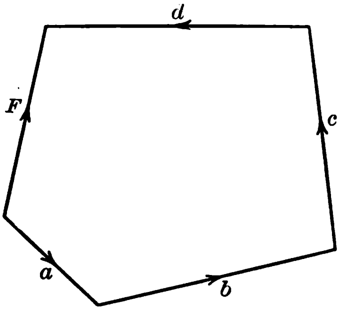


Fig. 13.

(ii) The Scalar Product of Two Vectors.

33. The second new feature in vector algebra is the introduction of a quantity, already much employed in physics, and known as "the *scalar product* of a pair of vectors." Such products are frequently indicated by round brackets as in Eq. 19; but where no confusion is caused, these will be omitted.

Let $\overline{A}$ and $\overline{B}$ be any two vectors whose included angle is θ, then their scalar product is defined by the following equation:

$$\text{Scalar product of } \overline{A} \text{ and } \overline{B} = (\overline{AB}) = AB \cos \theta. \qquad \text{Eq. 19.}$$

Thus if $\overline{A}$ be a force, and $\overline{B}$ a linear displacement, the scalar product represents the work done by the force A during the displacement B. Later on the student will find that the Principle of Virtual Displacements — one of the most important in Dynamics — is expressed as an equation between scalar products of the type defined in Eq. 19.

Interesting Special Case. Unit Vectors.

34. Suppose that the two vectors to be multiplied together are unit vectors. And first let us form the scalar product of each of the unit vectors with itself; then we shall obtain by Eq. 19,

$$i^2 = 1, \quad j^2 = 1, \quad \text{and} \quad k^2 = 1. \qquad \text{Eq. 20.}$$

It will thus be seen that in general the scalar square of any vector is the square of its tensor; thus,

$$\overline{F}^2 = F^2. \qquad \text{Eq. 21.}$$

By the same general principle, Eq. 19, it will be seen that when we select for our pair of vectors any two of the three unit vectors, $\overline{i}$, $\overline{j}$, and $\overline{k}$, we obtain

$$\overline{i}\,\overline{j} = 0, \quad \overline{j}\,\overline{k} = 0, \quad \text{and} \quad \overline{k}\,\overline{i} = 0. \qquad \text{Eq. 22.}$$

Parallelism of two vectors is indicated by the fact that their scalar product is the product of their tensors; thus,

$$\overline{F}\,\overline{H} = FH. \qquad \text{Eq. 23.}$$

The fact that two vectors are perpendicular to each other is indicated by the zero value of their scalar product; thus,

$$\overline{F}\,\overline{H} = 0. \qquad \text{Eq. 24.}$$

It is, of course, assumed in Eq. 24 that neither F nor H is zero. The angle between any two vectors is indicated by the scalar product of their corresponding unit vectors; thus,

$$\overline{i_f}\,\overline{i_h} = \cos\theta. \qquad \text{Eq. 25.}$$

Where ambiguity might otherwise exist the subscript will indicate the vector of which i is the unit.

It will be observed also that scalar multiplication is subject to the *commutative law* of ordinary algebra. Thus,

$$\overline{F}\,\overline{H} = \overline{H}\,\overline{F},$$

and, since the distributive law also holds, we have

$$\begin{aligned}\overline{F}\,\overline{H} &= (F_x\bar{i} + F_y\bar{j} + F_z\bar{k})\,(H_x\bar{i} + H_y\bar{j} + H_z\bar{k})\\ &= F_xH_x + F_yH_y + F_zH_z, \qquad \text{Eq. 26.}\end{aligned}$$

another exceedingly useful form of the scalar product.

(iii) The Vector Product.

35. The third new feature of vector algebra is the introduction of a quantity which is called the " vector product " of a pair of vectors; this is denoted by square brackets. The defining equation is as follows:

Vector product of $\overline{F}$ and $\overline{H} = [\overline{F}\,\overline{H}] = FH\,sin\,\theta = \overline{P}$, say, Eq. 27.

where θ is, as before, the angle included between the two vectors, *and where the direction of the product is perpendicular to the plane defined by the pair of vectors $\overline{F}$ and $\overline{H}$.* Further, if we denote this product by $\overline{P}$, then the sense of $\overline{P}$ is such that a right-handed rotation about $\overline{P}$ will carry the vector $\overline{F}$ to $\overline{H}$. The meaning of this product $\overline{P}$, as well as its direction and sense, will be clear from Fig. 14. From this definition of vector product, it is at once evident that the vector square of any vector is zero, since in this case θ is zero, and hence also θ is zero. If now we apply the defining equation of

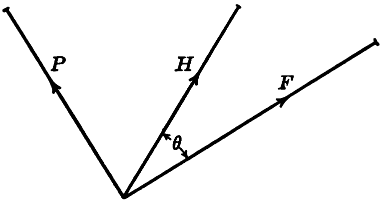


Fig. 14.

the vector product to the three unit vectors $\bar{i}$, $\bar{j}$, and $\bar{k}$, we shall obtain

$$[\bar{i}\,\bar{i}] = 0, \quad [\bar{j}\,\bar{j}] = 0, \quad [\bar{k}\,\bar{k}] = 0, \qquad \text{Eq. 28.}$$

and

$$[\bar{i}\,\bar{j}] = \bar{k}, \quad [\bar{j}\,\bar{k}] = \bar{i}, \quad [\bar{k}\,\bar{i}] = \bar{j}. \qquad \text{Eq. 29.}$$

Since *sin* θ and *sin* $(-\theta)$ have opposite signs, it follows that when the order of multiplication is changed, the sign of the product is also changed. Thus,

$$\bar{P} = [\bar{F}\,\bar{H}] = [-\bar{H}\,\bar{F}]. \qquad \text{Eq. 30.}$$

In other words, the commutative law does *not* hold in the case of vector multiplication.

36. Later on (§ 67) it will be seen that the fundamental theorem of moments is most elegantly expressed as an equation between vector products of the type defined by Eq. 27.

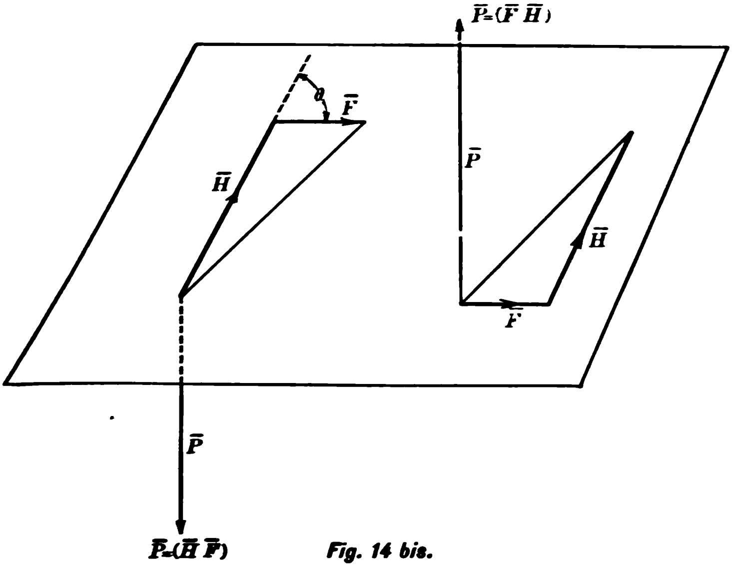


Fig. 14 bis.

The simplest method of getting a clear grasp of the geometrical meaning of Eq. 30 — the failure of the commutative law — is perhaps to consider the two moments represented in Fig. 14 *bis*, where, on the left-hand side, the force F is represented as acting through a lever arm H. The rotation is clockwise, about OP as axis.

In the right-hand side of the figure the vectors are the same in sense, direction, and amount; but the order of their application has been changed. F is now a lever arm, and H is a force. It will be observed, however, that the rotation is now counter-clockwise; in other words, the sign of the vector product has been changed by changing the order of multiplication. Observe the right-handed screw relation, and there will be no ambiguity in the sign of vector products.

Rectangular Components of Vector Product.

37. The vector product of $\bar{F}$ and $\bar{H}$ may also be written in terms of the rectangular components of $\bar{F}$ and $\bar{H}$ as follows:

$$\bar{P} = \bar{i}\,(F_yH_z - F_zH_y) + \bar{j}\,(F_zH_x - F_xH_z) + \bar{k}\,(F_xH_y - F_yH_x). \qquad \text{Eq. 31.}$$

To prove that the tensors of each side of this equation are equal, we have only to take the scalar square of each side, remembering that since the three components are mutually at right angles there are no products.

$$P^2 = (F_yH_z - F_zH_y)^2 + (F_zH_x - F_xH_z)^2 + (F_xH_y - F_yH_x)^2.$$

Now by ordinary scalar algebra this becomes

$$\begin{aligned} P^2 &= (F_x^2 + F_y^2 + F_z^2)\,(H_x^2 + H_y^2 + H_z^2) \\ &\quad - (F_xH_x + F_yH_y + F_zH_z)^2 \\ &= F^2H^2 - F^2H^2\cos^2\theta \\ &= (FH\sin\theta)^2, \end{aligned}$$

or

$$P = FH\sin\theta,$$

which proves that the tensors in Eqs. 27 and 31 are equal. The next step is to prove that the direction of $\overline{P}$ in Eq. 31 is at right angles to both $\overline{F}$ and $\overline{H}$. And this is easily done by multiplying the value of P as given in Eq. 31 by $\overline{F}$ and $\overline{H}$ respectively, which leads to $\overline{F}\,\overline{P} = 0$ and $\overline{H}\,\overline{P} = 0$ respectively, and which proves that P is perpendicular to both F and H.

Hence P, as defined by Eq. 31, is identical with P as defined by Eq. 27.

It is left as an exercise for the student to prove that the distributive law holds in the case of vector multiplication.

Multiply together the following expressions for $\overline{F}$ and $\overline{H}$ respectively, and interpret the result according to Eqs. 28 and 29.

$$\overline{F} = iF_x + jF_y + kF_z.$$
$$\overline{H} = iH_x + jH_y + kH_z.$$

The student who is familiar with determinants will be interested in proving that the vector product may also be written in the following form:

$$[\overline{F}\,\overline{H}] = \begin{vmatrix} i & j & k \\ F_x & F_y & F_z \\ H_x & H_y & H_z \end{vmatrix}$$

Returning, from this digression, to kinematics proper, we are now equipped for consideration of

The Vector, Velocity, in Terms of the Scalar, Speed.

38. The position of a particle, being defined by a vector, may change in either one of two ways: namely, either in distance or direction. In either case the particle moves along a line, it may be straight, it may be curved; and for this reason a particle is said to move always with a *linear* velocity.

Let $\bar{r}$ be the position vector of the particle, $\bar{v}$ its linear velocity, and s the distance it has traveled along its path; then, since velocity is merely the rate of change of position, we have

$$\bar{v} = \frac{d\bar{r}}{dt} = \frac{d\bar{r}}{ds} \cdot \frac{ds}{dt}, \qquad \text{Eq. 32.}$$

where $\frac{ds}{dt}$ is the speed of the particle, and $\frac{d\bar{r}}{ds}$ is a unit vector drawn in the direction of the motion.

From Fig. 15 it will be clear that the value of $\frac{d\bar{r}}{ds}$ is unity. For the vector $d\bar{r}$ which must be added to the original vector, $\bar{r}$, to produce the new position vector, $\bar{r} + d\bar{r}$, is numerically equal to the increment of path ds. Accordingly, the value of $\frac{d\bar{r}}{ds}$, in the limit, is unity. The direction of $d\bar{r}$ will, of course, be tangent to the path at the point where the particle is at any instant t. Let us denote this unit vector by $\bar{i}_s$, then

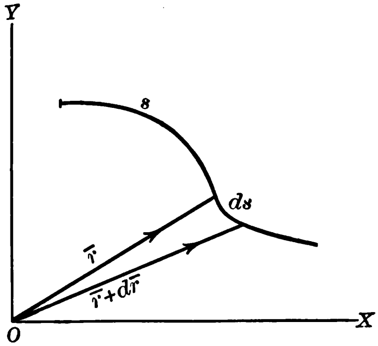


Fig. 15.

$$\bar{v} = \bar{i}_s \frac{ds}{dt}. \qquad \text{Eq. 33.}$$

We have then, in Eq. 33, a perfectly general expression for any linear velocity, as the product of two factors, one of which defines the speed, the other the direction of the motion.

Analogue. Rate of Change of Angular Position. Angular Velocity.

39. The position of a line in any given plane at any given instant is defined by the angle which this line makes with some fixed reference-line in that plane. The rate at which this angle varies with time is called the *angular speed* of the rotating line. Angles are sometimes measured in degrees, sometimes in radians, sometimes in complete revolutions.

$$1 \text{ revolution} = 360° = 2\pi \text{ radians}.$$

These three units, being all of the same kind and differing only in size, will be used as circumstances demand. Ordinarily the radian will be employed. When any other unit is introduced, this change will be clear from the context. Let us denote by θ the angle which the given line makes with the reference line; then the defining equation for angular speed, which we shall denote by ω, is as follows:

$$\omega = \frac{d\theta}{dt} . \qquad \text{Eq. 34.}$$

The position of one line with respect to another is, however, not completely determined by θ; one must add to this description, the direction of the normal to the plane in which θ is measured. In other words, the *axis* of rotation must be given as well as the angle of rotation, in order to determine the angular position of a line. The rate at which the angular position of a line varies with time is called its *angular velocity*. Let us denote the angular velocity of a line by $\bar{\omega}$, then its defining equation becomes

$$\bar{\omega} = \frac{d\,\bar{\theta}}{dt} , \qquad \text{Eq. 35.}$$

where $\bar{\theta}$ is a vector, as described in Sec. 17.

The difference between angular speed and angular velocity is, then, that the latter takes into account the axis of rotation as well as the rate of rotation. The former is a scalar quantity; the latter, a vector. The two are strictly analogous, therefore, to linear speed and linear velocity. Angular speed merely tells us how rapidly a line is rotating; angular velocity tells us also the axis about which the rotation is taking place.

Vectorial Representation of Angular Velocity.

40. The angular displacement of any line is most easily represented by a vector drawn along the axis of rotation, such that its length is numerically equal to the number of units of angle contained in the angular displacement. And this vector is always drawn in such a sense that it will be related to the angular displacement (rotation) in the same way that the advance of a nut on a right-handed screw is related to the rotation of the nut. See Fig. 16. For brevity, we shall hereafter allude to this as "the right-handed screw relation."

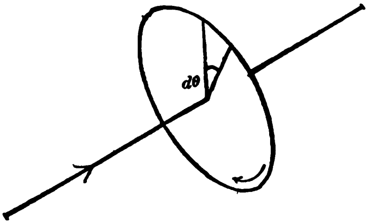


Fig. 16.

Angular velocity is represented in the same manner as angular displacement, viz., by a vector drawn normal to the plane of rotation. *The length of this vector measures the angular speed, and the direction of the vector is the direction of the angular velocity.* Let us denote by $\bar{i}_\theta$ a unit vector drawn in the direction of the axis of rotation. Then our final expression for angular velocity in terms of angular speed becomes

$$\bar{\omega} = \bar{i}_\theta \frac{d\theta}{dt}, \qquad \text{Eq. 36.}$$

an exact analogue of

$$\bar{v} = \bar{i}_s \frac{ds}{dt} . \qquad \text{Eq. 33.}$$

Relation between Angular and Linear Velocity.

41. Consider any particle on a pulley rotating about a fixed axis with an angular speed ω. Let the perpendicular distance of the particle from the axis of rotation be r. Then it is evident that we may view this particle either as part of a rigid body describing ω radians of arc each second, and traveling, therefore, through a distance of $r\omega$ centimeters each second, or we may think of the particle as traveling in a circular path of radius r with a linear speed v. Since these two linear velocities are numerically equivalent, we may write

$$v = r\omega. \qquad \text{Eq. 37.}$$

And since the direction of v is perpendicular to those of r and ω respectively, we may also write,

$$\bar{v} = [\bar{\omega}\,\bar{r}],$$

and thus obtain an equation connecting the analogous quantities $\bar{v}$ and $\bar{\omega}$. Later we shall meet other equations of this type, which we shall call *cross-over equations.*

42. We have seen (§ 21) that the most general motion of a rigid body is the resultant of a pure rotation and a pure translation. The cross-over, Eq. 37, tells us how we may pass from angular to linear velocities.

Accordingly, we may express the linear velocity of any point, say P, on a body by adding to the translational component $\bar{v}_0$, the linear velocity $[\bar{\omega}\,\bar{r}]$ due to the rotation. Thus in Fig. 16 *bis*, let O be *any* fixed point on the axis of rotation; the velocity of the point P whose radius vector $\overline{OP}$ is $\bar{r}$ will be $\bar{v}$ where

$$\bar{v} = \bar{v}_0 + [\bar{\omega}\,\bar{r}], \qquad \text{Eq. 37 } \textit{bis}.$$

and $\bar{v}$ is the geometrical sum of $\bar{v}_0$ and a velocity $r\omega \sin \phi$, having direction perpendicular to both $\bar{r}$ and $\bar{\omega}$.

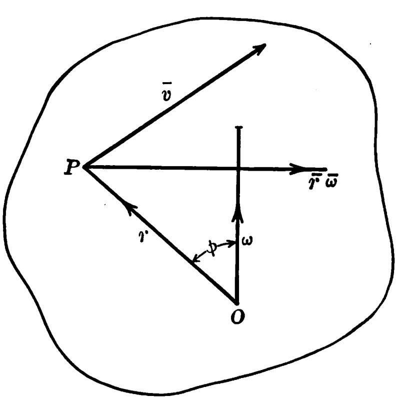


Fig. 16 bis.

It will be observed that the square brackets in the right-hand member are used to denote a vector product. It is important for the beginner to note what are the data of the problem, namely, $\bar{r}$ determined by the position of the point P, and v_0 and ω, the given velocities of the body. It should be noted also that the signs in this Eq. 37 *bis* are correct only when the right-handed system of coördinates are used.

PROBLEMS.

1. An automobile having a 28-inch wheel travels with a speed of 12 miles an hour. What is the angular speed of the wheel about the axle?

2. A belt travels over a 30-inch pulley which is carried on a shaft making 300 revolutions per minute (generally written 300 R.P.M.). Find the linear speed of the belt in feet per second.

3. The gear wheel on the stud (driving shaft) of a lathe has eighteen teeth on it, and meshes into a gear wheel of 48 teeth on the feed-screw. How does the angular speed of the feed-screw compare with that of the stud?

4. If in the preceding problems the feed-screw has a pitch of one-twelfth inch, what thread will the lathe cut with this pair of gears?

5. (*a*) A gear wheel with 8 teeth runs with a constant angular speed of 100 R.P.M. and meshes into another gear wheel having 24 teeth. Find the angular speed of the second wheel.

(*b*) A third gear having 48 teeth is now inserted *between* these two so as to make a train of 3 wheels. How will the insertion of this third gear affect the angular speed of the second wheel?

6. The fly-wheel of a gas engine has a diameter of 54 inches and an angular speed of 256 R.P.M. It is desired to use this engine to drive a dynamo armature at the rate of 900 R.P.M. What size of pulley must be employed on the shaft of the dynamo? Assume that there is no slipping of the belt.

Rate of Change of Velocity. Acceleration.

43. If all the velocities which we meet in nature were uniform, that is, constant both in direction and in speed, the problems of kinematics would be simple indeed. But practically all velocities are found to vary either in direction or in speed, or in both.

We shall have frequent occasion to consider the *rate* at which the velocity varies with the time.

What is meant by the time-rate of variation of velocity may perhaps be explained most clearly by a graphical method. Let the vector OP_1 denote the velocity of a particle at any instant t_1, and let the vector OP_2 be the velocity of the same particle at the instant t_2. Then the change of velocity during the interval $t_2 - t_1$ is represented, both in direction and in speed, by the vector $\overline{P_1P_2}$.

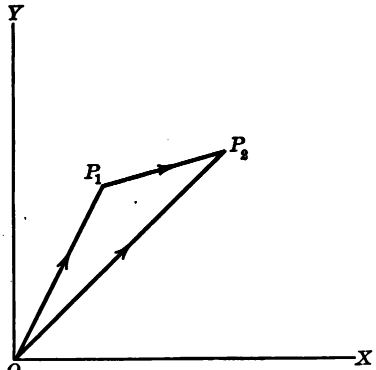


Fig. 17.

The ratio $\frac{\overline{P_1P_2}}{t_2-t_1}$ is the average rate at which the velocity of the particle is changing, and is called the *average acceleration*, during the interval.

In order to obtain the *acceleration* at any particular instant, we have merely to take the limit which this ratio approaches at that particular instant.

In this case the vector $\overline{P_1P_2}$ becomes infinitesimal; let us denote it by $d\bar{v}$; and instead of $t_2 - t_1$ write dt. Then the defining equation for acceleration at any instant becomes

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d^2\bar{r}}{dt^2}. \qquad \text{Eq. 38.}$$

The value of this differential coefficient, $\frac{d\bar{v}}{dt}$, is, at any instant, perfectly clear and definite. The numerator being a vector and the denominator a scalar, it is evident that the differential coefficient itself — that is, the acceleration — is a vector. The direction of the acceleration is, plainly, the direction of the increment $d\bar{v}$.

We have denoted the path of a particle by s, and the speed of the particle by $\frac{ds}{dt}$. It is very important for the student to observe that the acceleration of a particle is *not* in general equal to $\frac{d^2s}{dt^2}$. This differential coefficient represents only the rate at which the speed is changing. If the velocity of the particle were constant in direction, then, and then only, would $\frac{d^2s}{dt^2}$ be the entire acceleration. But in general $\frac{d^2s}{dt^2}$ is only that component of the acceleration which lies in the direction of the path, s.

44. The following simple and modern point of view introduced by users of vector analysis makes clear just how the total acceleration, $\frac{d\bar{v}}{dt}$, may be resolved into two components, one along and one at right angles to the path of the particle.

From Eq. 33, § 38, we have

$$\bar{v} = \bar{i}_s \frac{ds}{dt},$$

where $\bar{i}_s$ is a unit vector having the direction of the tangent to s at each instant.

By the ordinary rules for differentiation of a product, we have

$$\bar{a} = \frac{d\bar{v}}{dt} = \bar{i}_s \frac{d^2s}{dt^2} + \frac{d\bar{i}_s}{dt}\frac{ds}{dt}\ . \qquad \text{Eq. 39.}$$

The result is already complete; we have only to interpret the differential coefficient $\frac{d\bar{i}_s}{dt}$. Since $\bar{i}_s$ is a unit vector, its scalar value cannot change; the only thing about it that can vary with time is its direction; and the time variation of direction is an angular velocity. It follows, therefore, that $\frac{d\bar{i}_s}{dt}$ is an angular velocity. It remains only to determine the axis of this angular velocity.

Let us imagine a particle at P_1 (Fig. 18), at any instant t, and at P_2 at the instant $t + dt$.

Then the change in direction of the unit vector i_s at these two instants will be shown in the small figure to the right.

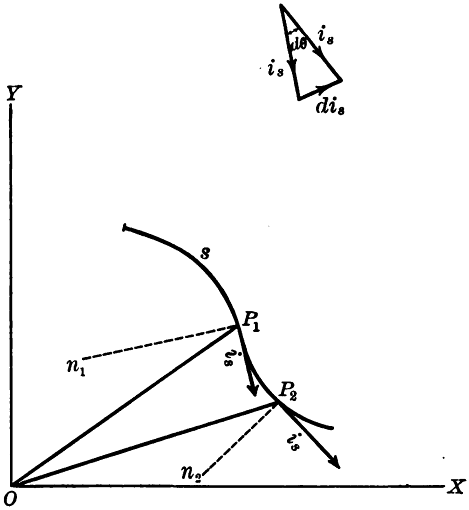


Fig. 18.

If $d\bar{\theta}$ denote the change of direction, we shall have, from the geometry of the small figure,

$$d\bar{i}_s = -\ [i_s d\bar{\theta}].$$

Dividing each side of this equation by dt, and passing to the limit, we have

$$\frac{d\bar{i}_s}{dt} = -\left[\bar{i}_s \frac{d\bar{\theta}}{dt}\right] = -\left[\bar{i}_s\, \bar{\omega}\right] = \left[\bar{\omega}\, \bar{i}_s\right]$$

where $\bar{\omega}$ is the angular velocity with which the tangential direction of the path is changing; the direction of $\bar{\omega}$ is therefore along the normal to the osculating plane; i.e., the plane containing the two tangents to the path at P_1 and P_2. The direction of $\bar{\omega}$ is therefore at right angles to $\bar{i}_s$; and hence the direction of the product $\bar{i}_s \bar{\omega}$ is, according to the rule for vector products, perpendicular to both $\bar{i}_s$ and $\bar{\omega}$. Accordingly, the direction of $\bar{i}_s \bar{\omega}$ lies in the osculating plane along the normal to the path of the particle. We may therefore write $[\bar{\omega}\,\bar{i}_s] = \bar{i}_n\,\omega$ where ω without the dash is a positive scalar quantity, and $\bar{i}_n$ is a unit vector drawn along the normal to the path toward center of curvature.

A simpler method of reaching this same result is to observe that the direction of $\frac{d\bar{i}_s}{dt}$ is completely determined by its numerator; this numerator is the increment di_s, in the small diagram to the right, in Fig. 18; and hence the direction of $\frac{d\bar{i}_s}{dt}$ is perpendicular to the tangent or normal to the path of the particle. Rewriting Eq. 39, we have then for the total acceleration of a particle moving along any path in any manner,

$$\bar{a} = \bar{i}_s \frac{d^2s}{dt^2} + \bar{i}_n\,\omega \frac{ds}{dt}, \qquad \text{Eq. 40.}$$

where $\frac{d^2s}{dt^2}$ measures the component of acceleration along the path, and $\omega \frac{ds}{dt}$ is the component of acceleration normal to the path. The first of these we may call the *tangential acceleration*; the second, the *normal acceleration*.

Translated into words, the tangential acceleration is the time rate of change of speed; the normal acceleration the product of the linear speed of the particle and the angular speed of the tangent to its path.

45. Using the cross-over equation 37, we have

$$\begin{aligned}\bar{a} &= \bar{i}_s \frac{d^2s}{dt^2} + \bar{i}_n \left(\frac{ds}{dt}\right)^2 \Big/ r = \bar{i}_s \frac{d^2s}{dt^2} + \bar{i}_n \frac{v^2}{r} \\ &= \bar{a}_s + \bar{a}_n,\end{aligned}$$

where the scalar values of $\bar{a}_s$ and $\bar{a}_n$ are

$$\left.\begin{aligned} a_s &= \frac{d^2s}{dt^2} \\ a_n &= \frac{v^2}{r} \end{aligned}\right\} \qquad \text{Eq. 41.}$$

and r is the radius of curvature at the point in question.

It is these scalar values, of course, that are always to be employed for purposes of numerical calculation. Since these two components are always at right angles to each other, one may obtain the scalar value of the total acceleration by squaring and adding thus:

$$a = \sqrt{a_s^2 + a_n^2} = \sqrt{\left(\frac{d^2s}{dt^2}\right)^2 + \left(\frac{v^2}{r}\right)^2}. \qquad \text{Eq. 42.}$$

THREE IMPORTANT SPECIAL CASES.

Case I. — Rectilinear Motion.

46. Suppose a particle to move in any manner whatsoever along a straight line, i.e., along a line of zero curvature. We shall then have $r = \infty$, and hence $\frac{v^2}{r} = 0$. Substituting this value in the preceding equation 42, we observe that the total acceleration is along the path of the particle, and its numerical value is defined by the following equation:

$$a = \frac{d^2s}{dt^2}. \qquad \text{Eq. 43.}$$

This case is well illustrated by any particle falling freely, *from rest,* under gravity. When the particle has an initial velocity which is not in a vertical direction, its path will be curved, and $\frac{d^2s}{dt^2}$ will no longer measure the *total* acceleration.

If, however, we denote the vertical displacement of a particle by y, and the constant acceleration of gravity by g, then a particle, say a bullet, may be projected in any direction, with any initial velocity, and we shall always have, after the bullet leaves the gun,

$$a = g = \frac{d^2y}{dt^2} .$$

The relation between $\frac{d^2s}{dt^2}$ and $\frac{d^2y}{dt^2}$ in the case of bodies falling freely, but not from rest, will be clear from the three vectors indicated in the accompanying figure. Expressed in words, the acceleration of gravity is the resultant of the two accelerations a_s and a_n, or, in vector notation,

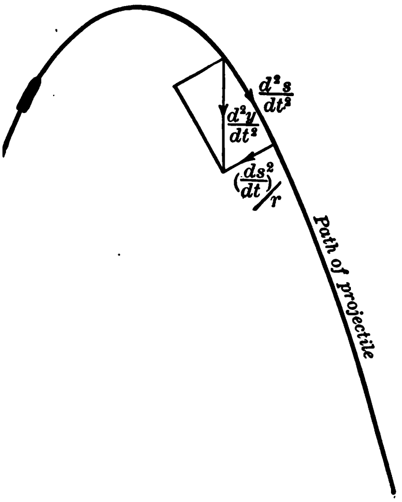


Fig. 19.

$$\frac{d^2y}{dt^2} = \dot{i}_s \frac{d^2s}{dt^2} + \dot{i}_n \frac{\left(\frac{ds}{dt}\right)^2}{r} .$$

47. Returning to Eq. 43, and integrating each side with respect to t, we obtain the

$$\text{speed} = \frac{ds}{dt} = at + v_0, \qquad \text{Eq. } 43\tfrac{1}{4}.$$

where v_0 is a constant of integration and denotes the speed at the instant $t = 0$.

Integrating again, one obtains

$$s = \tfrac{1}{2}\, at^2 + v_0 t + s_0, \qquad \text{Eq. } 43\tfrac{1}{2}.$$

where s_0 is a constant of integration denoting the initial position of the particle, and s the distance along the path from the origin.

Case II. — Uniform Circular Motion.

48. If a particle moves in a circular path with a constant speed, we may describe these circumstances by writing

$$r = \text{constant},$$

and

$$\frac{d^2s}{dt^2} = 0.$$

Accordingly, the total acceleration is perpendicular to the path of the particle (i.e., along the radius and toward the center), and its numerical value is defined by the following equation:

$$a = \frac{v^2}{r} = \omega v = r\omega^2. \qquad \text{Eq. } 44.$$

The time occupied by the particle in once describing the circumference of the circle is called the *period* of the motion. If we denote the period by T, then

$$\omega T = 2\pi, \text{ and } a = \frac{4\pi^2 r}{T^2} = \frac{2\,\pi v}{T}.$$

In any periodic motion, the reciprocal of the period, $\frac{1}{T}$, is generally called the *frequency*.

Case III. — Simple Harmonic Motion.

49. Let us now project this uniform circular motion upon any arbitrarily selected diameter of the circle; and let us denote distances measured along this diameter by x; then the acceleration along this diameter will be denoted by $\frac{d^2x}{dt^2}$; but since this is the projection of the total acceleration, at any instant, we have

$$-\frac{d^2x}{dt^2} = r\omega^2 \cos \theta = r\omega^2 \cos \omega t, \qquad \text{Eq. 45.}$$

where θ is the angle between the axis of X and the radius vector of the particle at the instant t. As one may observe from Fig. 20, the acceleration will be towards the left whenever $\cos \theta$ is positive, and towards the right whenever $\cos \theta$ is negative. Hence the opposition of sign in Eq. 45.

This differential, Eq. 45, describes the motion of a point H which is the projection of the particle P upon the diameter OX. As the particle P moves around the circle with uniform speed, the point H moves to and fro across the diameter with a motion which is called *Simple Harmonic*, and which is generally written S.H.M.

Linear S.H.M. is, therefore, a rectilinear motion. Let us suppose the angle θ (Fig. 20) to be measured in a counterclockwise direction from the reference line, OX, and let us begin to reckon time from the instant when $\theta = 0$, i.e., when H reaches the end of the diameter.

By integrating Eq. 45, under these conditions we may obtain the complete behavior of the point H. Thus,

$$\frac{dx}{dt} = -\int_0^t r\omega^2 \cos \omega t \cdot dt = - r\omega \sin \omega t$$

$$= \text{speed of point } H \text{ at any instant, } t.$$

Integrating again, we obtain

$$x = -\int_0^t r\omega \sin \omega t \cdot dt = -r\omega \int_0^t \sin \omega t \cdot dt = r \cos \omega t$$

$$= \text{displacement of point } H \text{ at any instant, } t, \qquad \text{Eq. 46.}$$

a result which might have been inferred immediately from Fig. 20. This equation of motion tells us just where the particle is in its path at the time t. Comparing it with Eq. 45, which is the differential equation of motion, we see that

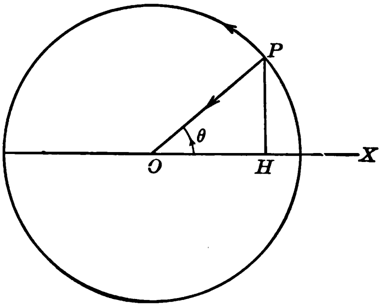


Fig. 20.

$$\frac{d^2x}{dt^2} = -\omega^2 x. \qquad \text{Eq. 47.}$$

Any motion represented by this equation is simple harmonic. The distance of the point H from the center of the circle, x, is called the *displacement*, and the maximum value of this displacement, r, is called the *amplitude* of the S.H.M.

Since ω^2 is constant and always positive in sign, it is evident from Eq. 47, that the acceleration is always proportional to, but opposite in sense to, the displacement. This proportionality is used as a *criterion* of S.H.M.

Remembering that $\omega T = 2\pi$, we may solve Eq. 47 for T, and obtain

$$T = 2\pi \sqrt{\frac{-x}{\frac{d^2x}{dt^2}}}, \qquad \text{Eq. 48.}$$

which enables us to write, at once, the period of any S.H.M. as soon as we know the constant ratio between displacement and acceleration.

Analogue. Rate of Change of Angular Velocity. Angular Acceleration.

50. The rate at which any angular velocity changes with time is called its *angular acceleration*. The difference between a good clock and a poor one is that in the former the angular acceleration of the hands is small; in the latter, large or irregular.

A steam engine is said to be well adapted to running a dynamo for incandescent lighting if the angular acceleration of its fly-wheel when once up to speed is small; otherwise the lamps will flicker.

Since we have already defined angular velocity by Eq. 36, § 40, we may obtain the angular acceleration at once by differentiation of this equation with respect to time, in a manner exactly analogous to that which gave linear acceleration by differentiation of linear velocity.

By definition,

$$\bar{\omega} = \bar{i}_\theta \frac{d\theta}{dt}, \qquad \text{Eq. 36.}$$

where ω is the total angular velocity, about the axis of spin, which we may call the axis of θ, and may be indicated by the subscript, θ.

For the sake of concreteness, we may think of a disk spinning with an angular speed about an axis θ which for the instant coincides in direction with the line which we have generally called the axis of X.

Now the angular velocity of this disk may change in either of two ways: (i) Its angular speed may change while the direction of the axis remains fixed. (ii) The direction of the axis may vary while the angular speed about the axle remains constant. Both of these changes may occur at the same time, which is, of course, the general case

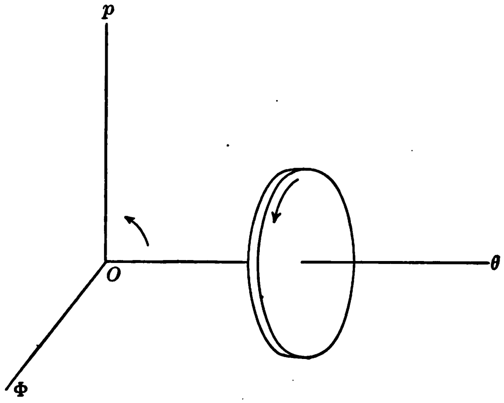


Fig. 21.

of angular acceleration. In order to obtain a quantitative expression for this angular acceleration in the general case, we have only to recall its definition, — rate of change of angular velocity, — and then, by the rule for differentiating a product,

$$\overline{A} = \frac{d\overline{\omega}}{dt} = \overline{i}_\theta \frac{d^2\theta}{dt^2} + \frac{d\overline{i}_\theta}{dt}\frac{d\theta}{dt}, \qquad \text{Eq. 49.}$$

where A is the total angular acceleration of the disk. The result is already complete; we have only to interpret the differential coefficient $\frac{d\overline{i}_\theta}{dt}$. Since $\overline{i}_\theta$ is a unit vector drawn

the axis of rotation, its scalar value cannot change;

thing about it that can vary with time is its direction; and the time variation of a direction is an angular velocity. It remains only to determine the axis of this angular velocity, $\frac{d\bar{i}_\theta}{dt}$, and its numerical value.

This is done in a manner exactly analogous to that employed in the linear case, § 44. In Fig. 22, let the two vectors marked $\bar{i}_\theta$ indicate two successive directions of the axis of rotation. Let the angle between them be $d\phi$. Then

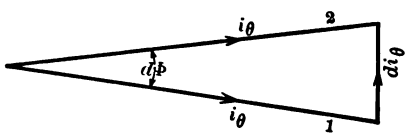


Fig. 22.

$$d\bar{i}_\theta = -[\bar{i}_\theta\, d\bar{\phi}].$$

Dividing each side of this equation by dt, and passing to the limit, we have

$$\frac{d\bar{i}_\theta}{dt} = -\left[\bar{i}_\theta \frac{d\bar{\phi}}{dt}\right] = -[\bar{i}_\theta\, \bar{\omega}_\phi] = [\bar{\omega}_\phi \bar{i}_\theta]$$

where ω_ϕ is the rate at which the axis of spin, θ, is changing direction; the direction of $\bar{\omega}_\phi$ is, therefore, along the normal to the plane which contains two successive positions of the axis of spin. The direction of $\bar{\omega}_\phi$ is, therefore, at right angles to i_θ; and hence the product $\bar{i}_\theta\, \bar{\omega}_\phi$ is, according to the rules for vector multiplication, a vector whose direction is at right angles both to $\bar{i}_\theta$ and $\bar{\omega}_\phi$. We may therefore write $\bar{i}_\theta \bar{\omega}_\phi = \bar{i}_p\Omega$ where the capital, Ω, is a scalar, and $\bar{i}_p$ is a unit vector drawn normal to the axis of spin, and also normal to the axis of the angular displacement, $d\phi$.

A simpler method of reaching this same result is to observe that the direction of $\frac{d\bar{i}_\theta}{dt}$ is completely determined by its numerator which is a vector; this numerator is the increment $d\bar{i}_\theta$ shown in Fig. 22, and hence the direction of $\frac{d\bar{i}_\theta}{dt}$ is, at any instant, perpendicular to the axis of spin, and perpendicular to the angular displacement $d\bar{\phi}$.

Rewriting Eq. 49, then we have for the total angular acceleration of any axis of spin,

$$\bar{A} = \bar{i}_\theta \frac{d^2\theta}{dt^2} + \bar{i}_p \Omega\omega, \qquad \text{Eq. 50.}$$

where $\frac{d^2\theta}{dt^2}$ measures the component of angular acceleration about the axis of spin, — let us call it the *axial component*,— and $\Omega\omega$ is the component at right angles to the preceding, which we may call the *normal component.* Again translating into words, the axial component is the rate of change of angular speed; the normal component is the product of the rate of spin by the rate of change of direction of spin.

Thus, in Fig. 21, if the axis of spin be rotated about the axis of ϕ, an angular acceleration will be produced about the axis of p.

An acceleration of this kind is a matter of great importance in the motion of spinning-tops, in the motion of mill shafting with one free end, in the motion of torpedo boats during maneuvers, and in similar problems.

For purposes of computation, one may write

$$\bar{A} = \bar{A}_\theta + \bar{A}_p,$$

where the scalar values of $\bar{A}_\theta$ and $\bar{A}_p$ are

$$\left.\begin{aligned} A_\theta &= \frac{d^2\theta}{dt^2} \\ A_p &= \Omega\omega \end{aligned}\right\} \qquad \text{Eq. 51.}$$

and

$$A = \sqrt{\left(\frac{d^2\theta}{dt^2}\right)^2 + (\Omega\omega)^2}, \qquad \text{Eq. 52.}$$

expressions which are strictly analogous to Eqs. 41 and 42.

Relation between Linear and Angular Acceleration.

51. Consider any particle which is moving in a circular path about the origin, say a particle on the driving pulley of a shaft.

The cross-over equation for velocity (37) gives us $v = r\omega$. The conditions of the problem give us $r =$ constant. Hence, by differentiation with respect to t, we have

$$\frac{dv}{dt} = r\frac{d\omega}{dt},$$

or

$$a = rA. \qquad \text{Eq. 53.}$$

This is the cross-over equation for acceleration in the case where the entire angular acceleration is axial, and applies to all cases of rotation about a fixed axis.

Analogue. THREE IMPORTANT SPECIAL CASES.

Case I. — Rotation About a Fixed Axis.

52. If the direction of the axis of spin does not change, we may describe this fact by writing, in Eq. 50, $\Omega = 0$; so that for the total angular acceleration, we have

$$\overline{A} = \overline{i}_\theta \frac{d^2\theta}{dt^2}, \qquad \text{Eq. 54.}$$

where $\overline{i}_\theta$ is a unit vector drawn along the fixed axis, and need not be explicitly indicated.

Integrating this equation once with respect to t, we have

$$\frac{d\theta}{dt} = At + \omega_0, \qquad \text{Eq. 55.}$$

where ω_0 is a constant of integration whose value is evidently that of the angular speed at the time $t = 0$. We may call it the initial angular speed.

This expression will be found very useful in computing the change of angular velocity, say in a fly-wheel or an armature, due to angular acceleration; and *vice versa*.

Integrating Eq. 55 once more, we have

$$\theta = \tfrac{1}{2} At^2 + \omega_0 t + \theta_0, \qquad \text{Eq. 56.}$$

where the constant of integration θ_0 is the initial value of the angular position θ. This expression will be found serviceable in getting the angular position of any rigid body turning on a fixed axis, and subject to acceleration such as that due to friction or to a clockweight.

Case II. — Rate of Spin Constant: Direction of Spin Changing at Uniform Rate.

53. These circumstances may be described by writing in Eq. 50,

$$\Omega = \text{constant},$$

and

$$\frac{d^2\theta}{dt^2} = 0.$$

Accordingly, the total acceleration is about an axis perpendicular to the axis of spin, and its numerical value is given by the following equation:

$$A = \Omega\omega. \qquad \text{Eq. } 56\tfrac{1}{2}.$$

The time occupied by the axis of spin in making one complete rotation about the axis normal to itself is called the *period of precession.*

If we denote this period by T_ϕ, then $\Omega T_\phi = 2\,\pi$, and

$$A = \frac{2\,\pi\omega}{T_\phi},$$

an expression which is strictly analogous to

$$a = \frac{2\,\pi v}{T},$$

as proved in §48. A motion of this kind is easily approximated in the ordinary gyroscope when the rotating wheel is not quite balanced. In Fig. 24, the arrows represent the case where the right-hand end of the horizontal axis is a little heavy.

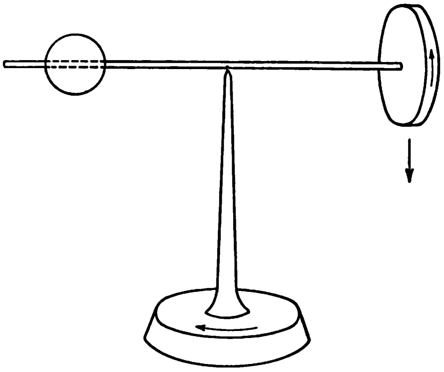


Fig. 24.

Case III. — Simple Harmonic Oscillation.

54. We now proceed to consider a special case under Case I, namely, rotation about a single fixed axis; but with the additional specification that the angular acceleration shall be proportional and opposite in sense to the angular displacement, a condition which we may express by writing

$$\frac{d^2\theta}{dt^2} = -a^2\theta, \qquad \text{Eq. 57.}$$

where a^2 is the constant of proportionality, and is numerically equal to the angular acceleration when $\theta = 1$.

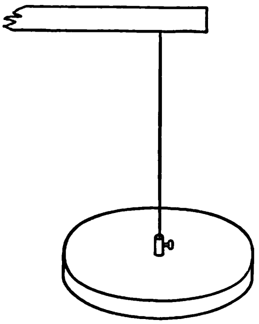


Fig. 25.

A motion of this kind is easily realized by clamping a metal disk to a metal wire as indicated in Fig. 25, and allowing it to rotate freely about the longitudinal axis of the wire. The balance wheel of a watch is another excellent example.

By differentiation of the following expression,

$$\theta = \Theta \cos (at + \varepsilon), \qquad \text{Eq. 58.}$$

it may at once be shown that it is a solution of Eq. 57 where Θ and ε are constants of integration. Evidently Θ is the largest value which θ can assume, and is called the *amplitude* of the oscillation. ε is the *epoch*; $at + \varepsilon$, the *phase*; and $\frac{2\pi}{a}$, the *period* of the motion. If we denote period by T, then we may write, as in linear simple harmonic motion,

$$T = \frac{2\pi}{a} = 2\pi \sqrt{\frac{-\theta}{\frac{d^2\theta}{dt^2}}}. \qquad \text{Eq. } 58\tfrac{1}{2}.$$

Summary of Vector Algebra.

(i) Vector expressed in terms of unit vector,

$$\overline{F} = \overline{i}_f F. \quad \text{or} \quad \overline{i}_f = \frac{\overline{F}}{F}.$$

(ii) Definition of scalar product,

$$\begin{aligned}(\overline{F}\,\overline{H}) = (\overline{H}\,\overline{F}) &= FH \cos \theta \\ &= F_xH_x + F_yH_y + F_zH_z.\end{aligned}$$

Scalar product of unit vectors, $\begin{cases}\overline{i}^2 = \overline{j}^2 = \overline{k}^2 = 1, \\ \overline{i}\,\overline{j} = \overline{j}\,\overline{k} = \overline{k}\,\overline{i} = 0.\end{cases}$

(iii) Definition of vector product,

$$[\overline{F}\,\overline{H}] = - [\overline{H}\,\overline{F}]$$

$$= \overline{i}\,(F_yH_z - F_zH_y) + \overline{j}\,(F_zH_x - F_xH_z) + \overline{k}\,(F_xH_y - F_yH_x)$$

$$= \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ F_x & F_y & F_z \\ H_x & H_y & H_z \end{vmatrix}$$

Scalar value of vector product $= FH \sin \theta$.

Vector product of unit vectors. $\begin{cases} [\bar{i}\,\bar{i}] = [\bar{j}\,\bar{j}] = [\bar{k}\,\bar{k}] = 0, \\ [\bar{i}\,\bar{j}] = \bar{k}, \ [\bar{j}\,\bar{k}] = \bar{i}, \ [\bar{k}\,\bar{i}] = \bar{j}. \end{cases}$

SUMMARY OF KINEMATICS.

Linear.	*Rotational.*
Coördinates x, y, z, pure translation.	Coördinates θ, ϕ, r, pure rotation.

Screw Motion.

Linear displacement, $d\bar{r}$.	Angular displacement, $d\bar{\theta}$.
Linear speed $= \frac{ds}{dt}$.	Angular speed $= \frac{d\theta}{dt}$.
Linear velocity $= \frac{d\bar{r}}{dt} = \bar{i}_s \frac{ds}{dt}$,	Angular velocity $= \frac{d\bar{\theta}}{dt} = \bar{i}_\theta \frac{d\theta}{dt}$,
$\bar{v} = \bar{i}_s \frac{ds}{dt}$.	$\bar{\omega} = \bar{i}_\theta \frac{d\theta}{dt}$.
Linear acceler't'n $= \frac{d\bar{v}}{dt} = \frac{d^2\bar{r}}{dt^2}$,	Angular acceler't'n $= \frac{d\bar{\omega}}{dt} = \frac{d^2\bar{\theta}}{dt^2}$,
$a = \bar{i}_s \frac{d^2 s}{dt^2} + \bar{i}_n \omega v$	$A = \bar{i}_\theta \frac{d^2\theta}{dt^2} + \bar{i}_p \Omega \omega$
$= \bar{a}_s + \bar{a}_n$.	$= \bar{A}_\theta + \bar{A}_p$.

Integral Formulae for Computation.

$a^2 = a_s^2 + a_n^2$.	$A^2 = A_\theta^2 + A_p^2$.
Case of Rectilinear Path.	*Case of Rotation about a Fixed Axis.*
$v = at + v_0$.	$\omega = At + \omega_0$.
$s = \frac{1}{2} at^2 + v_0 t + s_0$.	$\theta = \frac{1}{2} At^2 + \omega_0 t + \theta_0$.

Cross-over Equations.

$$v = r\omega.$$
$$a = rA.$$

ILLUSTRATIVE PROBLEMS.

Linear Acceleration.

1. A particle P has impressed upon it several simultaneous velocities v_1, v_2, v_3, etc., the direction cosines of the velocities being (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3), etc., respectively. Show that the components of the resultant velocity are given by the three following equations:

$$\frac{dx}{dt} = v_1 l_1 + v_2 l_2 + v_3 l_3 + \text{etc.} = \Sigma\,(v\,l),$$

$$\frac{dy}{dt} = v_1 m_1 + v_2 m_2 + v_3 m_3 + \text{etc.} = \Sigma\,(v\,m),$$

$$\frac{dz}{dt} = v_1 n_1 + v_2 n_2 + v_3 n_3 + \text{etc.} = \Sigma\,(v\,n).$$

2. Show that in the preceding problem the resultant velocity, v, may be expressed in either of the two following forms,

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2},$$

$$v = \frac{dx}{dt}\,l + \frac{dy}{dt}\,m + \frac{dz}{dt}\,n,$$

where (l, m, n) are the direction cosines of the resultant.

3. A bullet is fired into the air at some elevation above the horizon. Supposing it to describe a parabolic path, at what point in the path will the normal acceleration a_n (§ 44) be equal to the total acceleration a?

4. The speed of a particle moving in a tortuous curve is given by the following equation:

$$v = 3\,t + 5\,t^2,$$

where t is the time elapsed since the beginning of the motion. Find the tangential acceleration, a_s, at the end of 7 seconds.

5. A particle moves in a circle with a uniform speed of 8 cm. per sec., while the radius vector which joins the particle to the center of the circle rotates with an angular speed of 2 radians per second.

Find the normal acceleration of the particle and the radius of the circle.

6. Let us suppose the moon to revolve about the earth — as it very nearly does — with uniform speed in a circular orbit whose radius is 240,000 miles. Using the centimeter and second as units of length and time, the acceleration at the moon due to the earth is approximately $\frac{1}{4}$. Find the time required for the moon to make one revolution about the earth.

7. A given S.H.M. has a period of $\frac{1}{3}$ second and an amplitude of 2 centimeters. Find the displacement at the time $t = \frac{1}{18}$ second. Assume that the displacement is equal to the amplitude at the instant $t = 0$.

8. A particle moves in a curved path with a normal acceleration of 4 and a tangential acceleration of 3. Find the total acceleration of the particle.

9. At a given instant the radius of curvature of the path of a particle is 40, and the speed of the particle is 8. Find the normal acceleration at this instant.

10. A bullet is fired from a rifle with such a horizontal velocity as to just clear the earth and revolve about the earth as a satellite. Find its period; also its velocity. Assume the acceleration due to the earth as 980.

Analogue. Angular Acceleration.

11. The fly-wheel of a large engine runs with a uniform angular speed of 120 R.P.M. When steam is shut off, the friction of the bearings and of the air produces an angular acceleration of $\frac{1}{300}$ *radian per second per second.* How long will it take the wheel to come to rest?

12. In problem 11 how many revolutions will the fly-wheel make in stopping?

13. The shaft of a screw-propelled vessel rotates with a constant angular speed of 80 R.P.M. while the vessel steams with constant speed in a circle about a fixed buoy once every 2 minutes. Find the angular acceleration of the shaft both in direction and amount,

14. An ordinary gyroscope is spinning, at the constant rate of 6 rotations per second, about a horizontal axis. At the same time, it is precessing (after the manner of a top), at the rate of 2 rotations per second, about a vertical axis. Find the angular acceleration of a radius of the gyroscope wheel.

15. A circular disk rotates on a horizontal axis with an angular speed given by the following expression,

$$\omega = 2 + 3\,t + t^2 \text{ radians per second,}$$

while the horizontal axis rotates in a horizontal plane at the rate of 2 radians per second.

Show that the axial component of the total angular acceleration of the disk is $3 + 2\,t$ radians per second per second.

Find the normal component ($\Omega\omega$) of the angular acceleration at the end of the second second.

Compute the total angular acceleration at the end of the *third* second.

16. A carriage is drawn at such a speed that the rear wheel makes 30 revolutions per minute on its axle. The radius of the wheel is $\frac{150}{\pi}$ centimeters. Where is the instantaneous axis of rotation, and what is the angular speed of the wheel about the instantaneous axis?

17. Compute the angular velocity of the earth about its axis of rotation, and resolve this along a diameter of the earth passing through the 30th parallel of latitude.

18. A ball on a bowling alley is at a given instant spinning about a vertical axis in a clockwise direction at the rate of 40 radians per second, and at the same instant spinning about an E. and W. axis at a rate of 10 radians per second as the ball rolls towards the north. Find the resultant angular velocity.

CHAPTER II.

KINETICS.

55. Up to this point we have been concerned only with the mode in which certain motions are most easily and accurately described, and have not considered the sources or relationships of these motions.

But, as everyone knows from his early childhood, the motion of a body, in general, depends to a very large extent upon the amount of matter contained in it. The ease with which a team of given strength will start or stop a wagon depends in a large degree upon the amount of load in the wagon box.

Two balls may have the same size and shape, and may be traveling with the same velocity; but if one of them contains more matter than the other, it will be correspondingly more difficult to stop.

To quantify this dependence of motion upon matter, and to put the results in such a form that they will accurately describe our experience, is the object of the science of dynamics, to which kinematics is merely a preparatory study.

THE IDEA OF MASS.

56. One of the best methods for getting a clear grasp of dynamics is to take a few simple experiments and consider them thoroughly. The following one illustrates in an excellent manner the fundamental ideas of our subject:

Suspend two similar cans A and B by means of long threads from the ceiling. Put into one of these cans, say A, a piece of lead or some other heavy object. Compress a spring between the two cans, and tie them together with a single loop of string, as shown in Fig. 26. If now this string be burned with a match, the two cans, formerly at rest, will fly apart, B moving to the right, and A to the left. It will be observed, however, that the empty can acquires a much higher speed than the loaded one, although the spring acts upon each of them for the same length of time; in other words, the acceleration thus produced in the light can is much greater than the acceleration produced in the heavy can. We shall now suppose that, by some means, we have *measured* accurately the speeds with which A and B respectively leave the spring. The ratio between these two speeds will be the ratio between the accelerations in A and B respectively at each instant while the spring is forcing them apart.

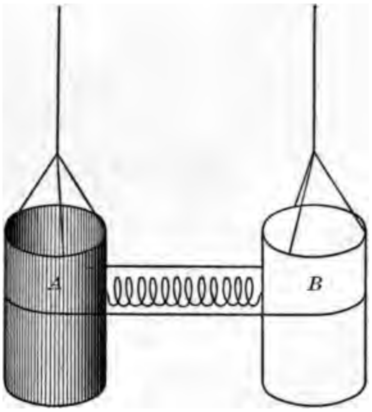


Fig. 26.

In every experiment of this kind it has been found that the more matter the vessel A contains in comparison with the vessel B, the less is its acceleration in comparison with that of B. Accordingly, the scientific world has agreed to take the ratio of these two accelerations, which we shall call a_A and a_B, as the measure of the inverse ratio of the amount of matter in A to that in B. This convention of the scientific world is expressed most neatly by denoting the amount of matter in A by M_A, that in B by M_B, and then writing

$$\frac{M_A}{M_B} = \frac{a_B}{a_A}. \qquad \text{Eq. 59.}$$

The quantity of matter in a body, which we have denoted by M, is more frequently called the *mass* of the body. As all our experience teaches, the difficulty of setting bodies in motion is proportional to their mass; likewise the difficulty in stopping bodies once in motion is directly proportional to their mass. Mass is, therefore, the measure of inertia, and *vice versa.* Mass, inertia, and quantity of matter as used in mechanics are indeed three synonyms with exactly equivalent meanings.

But it will be observed that Eq. 59 does not enable us to determine the value of either M_A or M_B. It gives us merely the ratio of these two masses. It becomes necessary accordingly to choose, arbitrarily, some one mass as a unit, and to refer all other masses to it by comparison, either in the manner above indicated, or in some other manner which is strictly equivalent to it.

The mass which has thus been selected as a standard is a piece of platinum which is kept at the International Bureau of Weights and Measures near Paris. It is known as the Standard Kilogram, or the *Kilogramme des Archives.*

All that is meant, therefore, by such operations as "determining the mass of a body," "measuring the inertia of a body," and "finding the amount of matter in a body," is a comparison of the mass of the given body with the mass of the standard body. As is well known, this operation is most accurately performed by means of a beam balance.

From what has preceded, it will be evident that no vector quantities are involved in the determination of mass, and that it is, therefore, a scalar quantity.

Center of Mass. Moment of Mass.

57. If we consider any rigid body as made up of a large number of particles, m_1, m_2, etc., say M, each of unit mass, the total mass of the body will be M. And if we locate a point in the body by the coördinates $\bar{x}$, $\bar{y}$, $\bar{z}$, where these coördinates satisfy the following equations,

$$\begin{aligned} M\bar{x} &= m_1x_1 + m_2x_2 + m_3x_3 + \text{etc.} = \Sigma mx, \\ M\bar{y} &= m_1y_1 + m_2y_2 + m_3y_3 + \text{etc.} = \Sigma my, \qquad \text{Eq. 60.} \\ M\bar{z} &= m_1z_1 + m_2z_2 + m_3z_3 + \text{etc.} = \Sigma mz, \end{aligned}$$

we shall find that the point thus defined has some exceedingly useful and very remarkable properties. It is called the *Center of Mass*, and we shall see that in many problems in dynamics we may substitute for the actual body a heavy particle of mass M situated at this center of mass.

If the origin of coördinates be chosen at the center of mass, $\bar{x}$, $\bar{y}$, $\bar{z}$, that is, if $\bar{x} = \bar{y} = \bar{z} = 0$, it follows that

$$\Sigma mx = \Sigma my = \Sigma mz = 0. \qquad \text{Eq. 60.}'$$

The product of the mass of a particle by its perpendicular distance from any given plane is called the *Moment of the Mass* of the particle with respect to that plane. In like manner, the moment of a particle with respect to an axis of rotation is the product of the mass of a particle by its perpendicular distance from the axis. In the same way, the moment of a particle with reference to a point is the product of the mass of the particle by the vector joining the particle to the point.

Accordingly, if we take the vector sum of all such products as $m\bar{r}$, where m is now the mass of the particle and $\bar{r}$ the position vector of the particle joining it to any fixed point arbitrarily chosen as origin, we shall obtain an expression $\Sigma m\bar{r}$, which is called the moment of the mass with respect to the given point.

If now we divide this product by the mass of the body M, we shall obtain a mean value of the vectors of all the particles of which the body is composed with respect to the given point, thus,

$$M\bar{R} = \Sigma m\bar{r},$$

or

$$\bar{R} = \frac{\Sigma m\bar{r}}{M}. \qquad \text{Eq. } 60''.$$

This is the defining equation for $\bar{R}$, which is the position vector of the center of mass. When the *given* point, the origin, is so chosen that $\bar{R}$ is zero, this point is the *Center of Mass* of the body, and is precisely the same as the point $(\bar{x}, \bar{y}, \bar{z})$ defined by Eq. 60.

It follows therefore from Eq. 60′ that the Center of Mass of any body is the point about which the Moment of Mass is zero. If we pass a plane through the Center of Mass, it does not follow that there are as many particles of unit mass on one side the plane as the other; but it *does* follow that the moment of mass of the particles on one side is the same as the moment of those on the other.

Summation by Integration.

58. In many cases it is very much more convenient to compute the position of the mass center according to the rules of the integral calculus.

Remembering the defining equation for density, we may write for m, in Eq. 60, $\rho\, dv$, where ρ is the volume density, and dv an element of volume.

Passing then to the notation of the integral calculus, we have as the equivalent of Eq. 60,

$$\bar{x} = \frac{\int_0^V \rho x\, dv}{\int_0^V \rho\, dv} = \frac{\int_0^V \rho x\, dx\, dy\, dz}{M},$$

$$\bar{y} = \frac{\int_0^V \rho y\, dv}{\int_0^V \rho\, dv} = \frac{\int_0^V \rho y\, dx\, dy\, dz}{M}, \qquad \text{Eq. } 60'''.$$

$$\bar{z} = \frac{\int_0^V \rho z\, dv}{\int_0^V \rho\, dv} = \frac{\int_0^V \rho z\, dx\, dy\, dz}{M}.$$

59. If the body whose mass center is sought is a plane lamina, the axes of X and Y may be chosen in this plane, and $\rho\, dz$ then represents the mass of unit area, a quantity usually called surface density.

Denoting this by σ, we have $\rho\, dz = \sigma$, and hence

$$\bar{x} = \frac{\int_0^S \sigma x\, ds}{\int_0^S \sigma\, ds} = \frac{\int_0^S \sigma x\, dx\, dy}{M},$$

$$\bar{y} = \frac{\int_0^S \sigma y\, ds}{\int_0^S \sigma\, ds} \quad \frac{\int_0^S \sigma y\, dx\, dy}{M}, \qquad \text{Eq. } 60^{\text{iv}}.$$

where ds is an element of surface.

When the surface is not plane, the center of mass will not, in general, lie in the surface, and its position must then be determined by the general expression, Eq. 60. Considerations of symmetry will, however, in many cases, greatly simplify the integration.

60. If the body be so nearly linear and straight that it may be treated as a straight line, then put

$$\sigma\, dy = \lambda = \text{mass per unit length} = \text{linear density},$$

and hence

$$\bar{x} = \frac{\int_0^L \lambda x\, dx}{\int_0^L \lambda\, dx} = \frac{\int_0^L \lambda x\, dx}{M}$$

where dx is an element of length of the linear body. Each of these three types of bodies will be illustrated by the following problems.

PROBLEMS.

1. A series of four heavy particles whose masses are 3, 2, 8, and 4 lie in a straight line at the following respective distances from the left-hand end of the line, 2 feet, 2½ feet, 3½ feet, and 4 feet. Find the mass center.

2. A circular iron disk is 3 inches in diameter. A circular hole 1 inch in diameter is punched out at such place that the center of the hole lies at a distance of 1 inch from the center of the disk. Find the center of mass of the remaining portion.

3. Prove that the center of mass of a triangular lamina lies on the median line ⅔ of the way from the vertex to the base.

4. Prove that the center of mass of any homogeneous right cone lies on the axis at a distance from the apex equal to ¾ of the height of the cone.

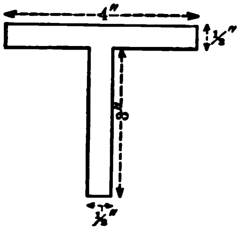


Fig. 27.

5. Find the mass center of the section of T-bar shown in Fig. 27.

6. Show that the center of mass of a straight wire whose linear density increases uniformly from one end to the other is analogous to that of a triangular lamina in that each mass center is distant from one end of the line ⅔ of its length.

7. A wire is bent into the shape of a circular arc of radius R. Choose the line which joins the center of the circle with the middle point of the arc as the axis of X, and the center of the circle as origin, then show that the mass center of the arc is given by the following equation,

$$x = \frac{\Sigma\, mx}{\Sigma\, m} = \frac{2\, R^2 \lambda \int_0^{\theta} \cos\theta\, d\theta}{2\, R\lambda \int_0^{\theta} d\theta} = \frac{R \sin\theta}{\theta},$$

where $2\,\theta$ is the angle subtended at the center by the arc, and λ is the linear density of the wire.

8. Using the result for a triangle, prove that the mass center of a circular sector has the same position as that of a circular arc of ⅔ the radius and of the same mass as the sector, and subtending the same angle as the sector.

9. Find the center of mass of a cross-section of a channel iron having the dimensions indicated in Fig. 28.

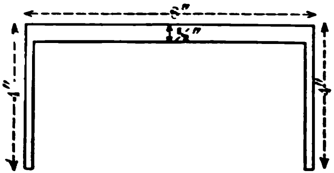


Fig. 28.

10. By direct integration determine the mass center of one quadrant of an elliptical lamina whose equation is

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

and whose surface density is 5.

DEFINITION OF MOMENTUM.

61. The principal mechanical quantities which we have considered up to this point are *position, velocity, acceleration,* and *mass.* The product obtained by multiplying the mass of a body by its linear velocity is a quantity so frequently employed in dynamics that a special name has been given to it, viz., *Linear Momentum,* which we shall call, when there is no danger of being misunderstood, simply *Momentum.*

Since velocity is a vector quantity and mass a scalar, it is evident that their product, momentum, is a vector quantity. If one is standing somewhere in front of a gun, the danger of his being injured depends quite as much upon the direction in which the bullet leaves the gun as upon the product of the mass and speed of the bullet.

In a certain very true sense, momentum represents the "*quantity of motion*" in any moving body. If, for instance, two freight cars are running free with the same velocity, one of them empty, the other loaded with pig-iron, it is evident that the latter car will do vastly more damage by collision, and that it may be said to have more "motion." In like manner, if two cars have equal masses, but move with different velocities, it is clear that we may fairly say that the car moving with higher speed has a greater "quantity of motion" than the other. It is in this sense that Newton uses the Latin word *motus* to denote quantity of motion. Hereafter we shall employ the word "momentum" whenever we wish to indicate quantity of motion, and shall define it by the following equation:

$$\textit{Linear momentum of any particle} = m\bar{v} \qquad \text{Eq. 61.}$$

where m is the mass, and $\bar{v}$ the velocity of the particle.

If it be required to find the momentum of a body of finite size, the most direct method is to integrate the product of $\bar{v}\,dM$ over the entire volume of the body. Here dM indicates the mass of an element of volume. Accordingly, the integral may be written,

Momentum of any body =

$$\int_0^M v\,dM = \int_0^V \rho v\,dx\,dy\,dz. \qquad \text{Eq. 62.}$$

Here, in general, ρ, the density, and $\bar{v}$, the velocity, will each depend upon the position of the element in the body; i.e., each will be a function of x, y, and z. If either of them is constant, the integration will be much simplified.

62. In the case of rigid bodies, a much simpler expression for momentum is obtainable in terms of the center of mass.

Recalling the expression for the velocity of any particle in a rigid body moving in the most general manner,

$$\bar{v} = \bar{v}_0 + [\bar{\omega}\,\bar{r}], \qquad \text{Eq. 37 } bis.$$

where $\bar{v}_0$ is the velocity of any point on the axis of rotation, we first obtain the momentum of a particle by substituting this value of $\bar{v}$ in Eq. 61, namely,

$$m\bar{v} = m\bar{v}_0 + m[\bar{\omega}\,\bar{r}] = m\bar{v}_0 + [\bar{\omega}\,.\,m\bar{r}],$$

since m is a scalar quantity.

Evidently the summation of this expression with respect to *all* the particles of the body will give us the momentum the entire body:

$$\Sigma m\bar{v} = \Sigma m\bar{v}_0 + \Sigma[\bar{\omega}\,.\,m\bar{r}].$$

Since $\bar{\omega}$ is constant for every part of a rigid body, and since $\bar{v}_0$ is the velocity of a particular particle, namely, that at the origin O, on the axis of rotation, we have

$$\begin{aligned} \Sigma m\bar{v} &= \bar{v}_0 \Sigma m - \bar{\omega}\ \Sigma m\bar{r} \\ &= \bar{v}_0 M - \bar{\omega}\ M\bar{R} && \text{by Eq. (60)}, \\ &= M(\bar{v}_0 - \bar{\omega}\,.\,\bar{R}) \\ &= M\bar{v}_c && \text{Eq. (63)} \end{aligned}$$

where $\bar{v}_c$ is the velocity of the center of mass, and M is, as before, the total mass of the body.

Another way of stating this theorem would be to say that the momentum of a body which has a *pure* rotation about an axis through the center of mass is zero. Or, in case of *any* rotation about an axis passing through the center of mass, $\bar{R}$ vanishes; and hence the entire linear momentum of the body, in this case, is due to its velocity of translation v_0 and is equal to $M\bar{v}_0$.

Analogue. THE IDEA OF ROTATIONAL INERTIA.

63. Let us suppose that we have two disks A and B mounted upon an axle and each provided with ball bearings. See Fig. 29. Imagine the disks to be connected by a spiral spring as shown in the diagram, so that a loop in each end of the spiral may be slipped over a pin in the adjoining face of the wheel. If now one disk be held fixed while the other is rotated in a direction opposite that of

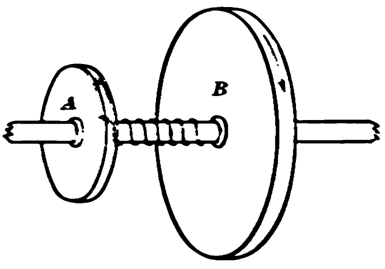


Fig. 29.

the arrow upon it, the spring will be wound up. The spring is so arranged that when it has unwound itself the loop slips off the pin and the disks run freely with whatever angular speed the spring may have impressed upon them. So much regarding the construction of this simple mechanism, which we shall suppose to be absolutely devoid of friction.

Let us now start with the spring wound up, and let both disks be released at the same instant. It is evident that the spring will act upon each of the two wheels with the same twisting effect and during the same length of time. But in general the angular acceleration which is thus impressed upon the two rotating bodies A and B will be different because the resistance which different bodies offer to any change in their angular velocity is different.

Let us call this resistance to change of angular velocity *Rotational Inertia,* and indicate it by I. It is possible then to define the ratio of the rotational inertias of these disks, $\frac{I_a}{I_b}$, by the following equation, which is strictly analogous to Eq. 59,

$$\frac{I_a}{I_b} = \frac{A_b}{A_a}, \qquad \text{Eq. 64.}$$

where A_a and A_b are the angular accelerations which the spring is observed to produce in the bodies A and B respectively.

64. In order to determine the rotational inertia of any body, two methods are now open to us. Either we can assume some standard body whose rotational inertia we arbitrarily call unity, and compare that of any other th this standard, or we may, by means of a cross-n, determine the rotational inertia in terms of its linear inertia and of its geometrical distribution. Arbitrary assumptions and values are strenuously avoided in science. The latter alternative has therefore been adopted, and the rotational inertia of a body is expressed in terms of its mass and its geometrical shape.

In the experiment represented in Fig. 29, it is clear that, since the spring acts equally and for the same length of time upon each of the two disks, we may write Eq. 64 in the following form,

$$\frac{I_a}{I_b} = \frac{A_b t}{A_a t} = \frac{\omega_b}{\omega_a}, \qquad \text{Eq. 65.}$$

where ω_a and ω_b are the *changes* in angular speed which the one spring produces in the two disks respectively.

In order to compare two rotational inertias, we have, therefore, only to compare the angular velocities which they acquire when each, *starting from rest,* is given the same twist for the same length of time by, say, a spiral spring.

Analogue. ANGULAR MOMENTUM.

65. The product of rotational inertia and angular velocity is a quantity of such frequent occurrence in dynamics that it has received a special name. It is called *angular momentum,* and is defined by the following equation:

$$\left.\begin{array}{r}\text{Angular Momentum of any rigid}\\ \text{body about any definite axis}\end{array}\right\} = I\omega, \qquad \text{Eq. 66.}$$

where I is the rotational inertia about that axis, and ω is the angular velocity about the same axis.

Just as linear momentum measures the amount of linear motion in a body, so angular momentum represents the angular motion in a body. This quantity is frequently employed under the name of "moment of momentum," — meaning thereby the moment of *linear* momentum. The reason for this will appear shortly when we find that $I\omega$ for any particle of mass, m, is equivalent to $m\overline{vr}$ (since $I = mr^2$ and $\omega = \frac{v}{r}\Big)$, r being the perpendicular distance from the axis of rotation to the direction of the velocity v.

ROTATIONAL INERTIA IN TERMS OF LINEAR INERTIA.

66. We now proceed to evaluate the quantity which we have called I in terms of its mass. In order to establish the cross-over equation connecting these two quantities, linear and rotational inertias, we shall appeal to some simple and crude experiments, asking the student to take for granted that the inferences drawn from them have been verified by thousands of more accurate tests and by thousands of inferences which observation has proved to be correct.

A couple of children are swinging on a gate; the gate is very much more easily set in motion when the children are near the hinge-line than when they are near the outer end of the gate. The hinge-line is the axis of rotation; the gate and the children constitute the rotating body; the linear inertia of the body is, therefore, the same wherever the children are located on the gate, but the rotational inertia increases as the children recede from the axis.

Two children who are see-sawing, one at each end of a board, know very well that the see-sawing goes on more smoothly and with greater uniformity when they are out near the ends of the board than when they are in near the middle. But when they are near the middle, the board is much more quickly started or stopped by a third party taking hold of the end. The mass of the board and the children is the same in each case; only the distribution of the mass is different.

The children and the board constitute a typical rotating system. The axis of rotation is the line of contact between the board and the log over which the board is balanced. The point to which attention is here directed is the fact that the resistance which a body offers to being set in rotation, i.e., its rotational inertia, is not measured by its mass merely, but depends also upon how far this mass is from the axis of rotation.

As all experience and accurate experiment show, the effectiveness of any given amount of momentum, mv, in keeping a wheel in rotation, depends upon the product of this momentum by the perpendicular distance of the momentum vector from the axis of rotation.

Suppose, for instance, that we have two fly-wheels in each of which the rotating mass is entirely concentrated in the rim of the wheel. Suppose that the rim of each wheel is moving *with the same linear speed* and that *the rims have equal masses*, so that mv, the *linear* momentum, is the same for each wheel. If the radius of one wheel is twice as large as the other, it will be found just twice as difficult to stop; or we may say that it has just twice the *angular* momentum of the other.

If now we assume what is perhaps not strictly justifiable from the preceding, but what is strictly correct and what will be later justified, namely, (1) that angular momentum measures the ability of a rotating rigid body to keep on rotating against a given resistance tending to stop the rotation, and (2) that the moment of linear momentum $\Sigma mv \,.\, r$ also measures the ability of a rotating rigid body to keep on rotating against the same resistance, then expressing these results in the form of an equation, we have

$$I\omega = \Sigma mv \cdot r = \Sigma mr^2 \cdot \omega, \qquad \text{Eq. 67.}$$

and hence

$$I = \Sigma mr^2 \text{ (Defining Eq. for } I\text{)} \qquad \text{Eq. 68.}$$

where the summation extends over all the particles in the body.

67. Having once established Eq. 67, the measure of angular momentum may be expressed very simply as follows:

DEFINITION OF A MOMENT.

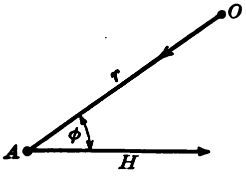


Fig. 30.

Having given any point O, fixed in space, and any vector quantity, say H, acting at any point A, the vector product of the position vector of A, with respect to O, and of the vector H is called the *moment* of H with respect to O. Thus, referring to Fig. 30,

$$[\bar{r}\bar{H}] = \text{moment of } H \text{ about } O = rH \sin\phi. \qquad \text{Eq. 69.}$$

In employing this equation, the usual conventions regarding sign are, of course, to be observed, namely, the rotation about O and the direction of the moment are related in the manner of the right-handed screw. The point A may be any point whatever in H.

Digression on Angular Momentum.

68. Employing the definition of moment just given, and the definition of I involved in Eq. 67, it becomes clear that *the Angular Momentum of any rigid body is the integral of* *the moment of linear momentum with respect to the axis of rotation.*

The most direct method of establishing Eq. 68 (which is really the cross-over equation between I and m) is by means of the energy expression which we shall meet a little later, § 115. Using the energy expression, I is introduced merely as a name for Σmr^2. Its physical interpretation must be sought later.

In the method here employed, we define angular momentum by analogy, and proceed to discover by experiment the proper *measure* for rotational inertia, or, what amounts to the same thing, the effective measure of linear momentum in maintaining rotation, namely, $[m\bar{v} \cdot \bar{r}]$. We have thus been led to substitute for our general and somewhat hazy notion of rotational inertia a definite quantitative idea which our later experience will prove to be thoroughly justifiable and useful. The proper name for this quantity is the "second moment of inertia," or "rotational inertia;" as a matter of fact, however, it goes by neither name, but is mostly called *Moment of Inertia*, a term introduced by Euler. Evidently it is an integral, obtained by integrating the expression $r^2\,dM$ over the entire mass.

$$I = \int_0^M r^2\,dM \quad \left\{\begin{matrix}\text{Defining Eq. for} \\ \text{Moment of Inertia.}\end{matrix}\right\} \qquad \text{Eq. 68}'.$$

where M is the mass of the entire body, and dM the mass contained in an infinitesimal volume whose distance from the axis is r. We shall perform these integrations for particular bodies when necessary. Since, however, the operation is a purely mathematical one, it is better to take the value of I from a table of integrals whenever possible.

69. We have now enlarged our vocabulary by the definition of four important and fundamental dynamical quantities, namely,

(1) Mass or Linear Inertia.
(2) Moment of Inertia or Rotational Inertia.
(3) Linear Momentum.
(4) Angular Momentum.

If these notions are clear, the way is open for an easy and accurate definition of *force* and *torque*.

FORCE AND TORQUE.

70. Few observations of the last three hundred years have been better established than the following, namely, whenever a body is left to itself, as free as possible from all external influences, it moves very nearly with uniform velocity, this velocity being that which the body has at the instant when it is left alone. As a matter of fact, it has been found to be practically impossible to "leave a body to itself," but the more nearly this condition is realized, the more nearly does the body move uniformly at the rate of motion which it had when external influences were removed.

Consider a perfectly level and smooth bowling alley; the ball (if not started too swiftly) rolls from one end to the other with approximately the same speed with which it leaves the hand of the bowler. If there were no resistance of the air and no rolling friction, we have every reason to believe that the ball would move the entire length of the alley with uniform velocity.

It is one of Galileo's chief merits that he discovered that a body set free from all external influence maintains a straight path with uniform speed. This was directly contrary to the Aristotelian doctrine, universally accepted at that period, that every body in motion requires the " cause of motion " to be continuously acting upon it. The new and radical view propounded by Galileo was that *change* of motion alone requires that the " cause of motion " should act.

The behavior of any body, of mass M, when freed from these external influences that are said to " cause " motion, may be described algebraically by saying that

$$Mv = constant. \qquad \text{Eq. 69.}$$

The fact expressed by this equation is generally called Newton's *First Law of Motion.* It is evident, of course, that the constant involved in Eq. 69 may have the value zero as well as other finite values; in other words, a state of motion and a state of rest are equally natural.

In like manner it has been observed that a body set in rotation and left to itself as nearly as possible will rotate with an angular velocity which is approximately constant. Thus if the steam be shut off from a large vertical turbine engine it is found that the cylinder will continue to rotate for something like four hours. If there were no friction at the bearings, it would doubtless run much longer; and if the air were entirely removed, the angular velocity would approximate constancy still more closely. In the limit we may assume, what we cannot prove experimentally but what never leads us astray, namely, that when a body is left to itself its angular momentum remains a constant. This fact also follows as an inference from Eq. 69. Algebraically we may express this as follows:

$$I\omega = \text{constant}. \qquad \text{Eq. 70.}$$

Summarizing, the linear and angular momenta of any body remain constant except in so far as these are compelled, by the action of other bodies, to change.

Definition of Force and Torque.

71. "The new idea appropriate to dynamics," as distinguished from kinematics, says Maxwell, "is that the motions of bodies are not independent of each other, but that under certain conditions dynamical transactions take place between two bodies, whereby the motions of both bodies are affected." . . . "We then confine our attention to one of the bodies and estimate the magnitude of the transaction between the bodies by its effect in changing the *momentum* of that body." . . . "The rate at which this change of momentum takes place is the numerical measure of the force acting on the body, and for all purposes of abstract dynamics it *is* the force acting on the body." *Nature,* XX, 214 (1879).

The *Force* acting upon any body is, therefore, defined as the rate at which the *linear* momentum of that body is changing.

In like manner the *Torque* (or Moment of Force) acting upon any body is the name given to the rate at which the *angular* momentum of the body is changing.

These names are used in this definite single sense throughout the entire field of physical science and engineering.

For purposes of computation we may express these definitions in the form of equations. Let us denote forces by F, and torques by L; then

$$\bar{F} = \frac{d}{dt}(M\bar{v}), \qquad \text{Eq. 71.}$$

$$\bar{L} = \frac{d}{dt}(I\bar{\omega}). \qquad \text{Eq. 72.}$$

The student who abides by these simple definitions of force and torque will find himself as free as possible from all such metaphysical considerations as the " cause of motion," " the tendency to produce motion," etc., — in terms of which force is often defined. Indeed, it is on account of our ignorance of the true physical cause of motion that science has been compelled to introduce this idea of force, which is a mere fiction, to be sure, but which is related to moving bodies (which are not fictions) in a definite, observable, and measurable way; namely, in the manner described by equations 71 and 72. Helmholtz puts the matter as follows: " In forming the conception of force, there is a danger of becoming entangled in empty tautology. Motions and accelerations are facts which can be observed; they are quantities whose values can be numerically determined by measurements of distances and times. When, however, forces are introduced as the causes of these motions, nothing is added to the knowledge of them which we have already obtained by observing the bodies in motion and which has already been expressed in terms of acceleration. Concerning force, one is unable, therefore, to assert anything which he has not already learned from the acceleration; and the introduction of this unexplained abstraction would therefore be in vain.

"The correct view — the one which justifies the introduction of the idea of force — is that which regards force as if it were a persistent cause, acting according to invariable laws, and always producing the same effect under the same circumstances. This is an attribute which acceleration does not possess. Thus we often meet cases in which the presence of a force must be assumed, but in which no acceleration appears. In these cases, we are not to assume that the force has ceased to act, but we must rather seek for *other* forces whose accelerations are such that when added to the missing acceleration they will give a vector sum which is zero." Helmholtz, *Vorlesungen*, Bd. I, Abth. 2, p. 24 (1898).

One may therefore summarize the views of Maxwell and Helmholtz by saying that while forces produce other effects than mass-accelerations, such, for instance, as the stretching of a spring or the bending of an elastic rod, they are nevertheless all measured in terms of a mass-acceleration as a standard.

The idea of a " cause " back of each force-effect properly remains in the mind; but this subject does not belong in the science of mechanics; indeed, the term " force " was introduced in order to avoid any such discussion.

Force and Torque Vector-Quantities.

72. Returning now to the consideration of the two expressions which define force and torque, Eqs. 71 and 72, it will be clear, since v and ω are each vectors while M, I, and t are each scalars, that $\overline{F}$ and $\overline{L}$ are each vectors, and are therefore to be resolved and compounded according to the general rules of vector algebra.

Action and Reaction.

73. Let us now recur to the general case of one body acting upon another, the " dynamical transaction " to which Maxwell alludes. We distinctly concentrated our attention upon one body, leaving the other out of consideration. But if we had just interchanged our bodies and had directed our attention to the other one, we should have found the time rate of change of momentum the same for it as for the first one.

If it were not so that, in a system of two bodies, the force which the first exerts upon the second is exactly equal to the force which the second exerts upon the first, then we should find the momentum of such a system changed by its internal forces alone, i.e. by the mutual action and reaction of its own parts.

Now as a matter of fact, no such change of momentum has ever been observed. Let us call the force of the body A upon the body B, F_{ab}; and the force of B upon A, F_{ba}; then the above result may be put into the following form:

$$F_{ab} = F_{ba}. \qquad \text{Eq. 73.}$$

In like manner, if A exerts a torque upon B, and if the system is composed of these two bodies only, then

$$L_{ab} = L_{ba}. \qquad \text{Eq. 74.}$$

The forces F_{ab} and F_{ba} are often called "action" and "reaction" respectively. In like manner L_{ab} and L_{ba} are often spoken of in the same terms.

74. Before proceeding further with the discussion of force, we shall here give the exquisitely accurate and brief, yet complete, summary first made by Newton, which describes implicitly, at least, all the observations and definitions which we have given up to this point.

NEWTON'S LAW OF MOTION.

LAW I. *Every body perseveres in its state of rest or of moving uniformly in a straight line except in so far as it is made to change that state by external forces.*

LAW II. *Change of linear momentum is proportional to the impressed force, and takes place in the direction in which the force is impressed.*

Law III. *Reaction is always equal and opposite to action; that is to say, the actions of two bodies upon each other are always equal and in opposite senses along the same direction.*

It is important for the student to observe that these laws are perfectly general; that is, they are true, so far as known, without any exception whatever. They are true for bodies at rest as well as for bodies in motion; they are true for fluids as well as for solids; they are true for a body of any size, shape, or composition.

75. This summary of Newton's was made especially with reference to translation; but, as will be evident both from experience and from the cross-over equations, analogous statements may be made for the rotation of rigid bodies.

Law I. *Every body perseveres in its state of rest or of uniform rotation about an axis which is constant in direction except in so far as it is made to change that state by external torques.*

Law II. *Change of angular momentum is proportional to the impressed torque, and takes place about an axis which is the same in direction as that of the torque.*

Law III. *Reaction is always equal and opposite to action; that is to say, the actions of two bodies upon each other are always equal and in opposite senses about the same axis.*

The remarks following Law III above hold equally for rotation and translation.

Relations of Newton's Laws to Each Other.

76. The first law describes the behavior of a body when there is no external force acting upon it; or, if you prefer, it states the conditions under which there are no external forces. The second law describes the behavior of bodies when external forces are acting. It is clear, therefore, that the first law is merely a special case of the second law, telling us what happens when the external forces have a particular value; namely, zero.

To clearly grasp the relation of the third law to the other two, it is helpful to recur to the idea of Maxwell, that when two bodies interact there is a "dynamical transaction" taking place between them. "Just as in commercial affairs the same transaction between two parties is called buying when we consider one party, selling when we consider the other, and trade when we take both parties into consideration." *Matter and Motion*, Art. 39.

The third law tells us, then, what happens when we include in our system all the bodies whose action we are considering. Under these conditions, — that is, under no external forces,—the third law says that the forces remaining are equal. The "forces remaining" may be called *internal* forces.

Put in other words, the first law furnishes one, but not the only, criterion for the presence (or absence, if you like) of force. The second law tells us how to measure force, — an idea due to Galileo (1564–1642). The third law expresses the fact that forces always occur in pairs which are equal and opposite.

The Criterion of Force.

77. The engineer's point of view is perhaps fairly represented by the following extract:

" In speaking first of force, we said that we measured and compared the magnitudes of forces only by the accelerations they produced or could produce. We find now that a body may be acted on by forces to any extent and yet be undergoing no acceleration. There is here, however, nothing contradictory. The conclusion which we have to draw when we see forces acting on a body which stands still, or moves with constant velocity, is that the accelerations produced by those forces must be such as exactly to counteract each other, so that as regards acceleration the state of the body is the same as if no force were acting at all. It is because we know that in such a case the body receives no acceleration that we say that the forces acting on it must be such as to produce accelerations whose sum is zero, and then infer that the sum of the forces must be zero also, because forces are proportional to the accelerations which they produce on the same body. Or otherwise, as we have already put it, if a body is visibly moving with uniform velocity, but is visibly also acted on by forces, we can only conclude that each force is producing its own acceleration, but that the forces are such that their whole accelerations exactly cancel each other. We should infer that if in such a case we took away any one of the forces, the body would at once begin to move faster or slower at a rate exactly corresponding to the now unbalanced part of the acceleration naturally due to all the remaining forces." Kennedy, *Mechanics of Machinery*, page 267 (4th ed.).

General Equations of Motion for a Particle.

78. The second law of motion enables us to write at once the equations of motion for a particle.

Let the mass of the particle be m, its coördinates x, y, z, and the components of its velocity, u, v, and w. Suppose that the resultant of all the external forces (i.e., all the forces except the reaction of the mass m) acting upon the particle has components X, Y, Z. Then the equations of translations are

$$m\frac{du}{dt} = m\frac{d^2x}{dt^2} = X,$$

$$m\frac{dv}{dt} = m\frac{d^2y}{dt^2} = Y, \qquad \text{Eq. 75.}$$

$$m\frac{dw}{dt} = m\frac{d^2z}{dt^2} = Z.$$

79. In passing now to the equations of rotation for a single particle, we have first to recall that rotational inertia has a definite meaning and a definite value *only with reference to some particular axis.* Rotational inertia is not a function of linear inertia alone; and hence, in rotation, there can be nothing corresponding to the conservation of mass. But if an axis of rotation is once fixed, and the particle is rigidly connected to this axis by means of a massless frame, we might then describe this motion in the following manner: Let θ denote the angular position of the particle P with reference to some plane of reference passing through the axis AB. Then if ω be the angular velocity of the frame about the axis, A the angular acceleration, and I the rotational inertia of the particle with reference to this same axis, then the torque L will be given by

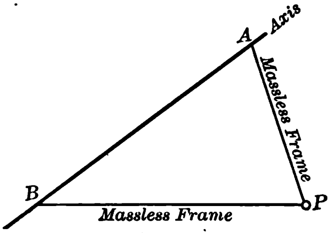


Fig. 31.

$$IA = I\frac{d\omega}{dt} = I\frac{d^2\theta}{dt^2} = L, \qquad \text{Eq. 76.}$$

an equation which is strictly analogous to any one of the three equations (75).

But a massless frame of the kind supposed above is not to be realized in nature; a little later we shall see, however, that a rigid body may be considered as a collection of heavy particles (the sum of whose masses is equal to the mass of the body), held together by a rigid massless frame, which, for want of a better name, we call cohesion.

DIGRESSION ON MOMENT OF FORCE.

80. In the meantime it may be rigorously proved that the moment of linear momentum for a particle of this kind is the angular momentum whose differential coefficient with respect to time is what we have called torque. No fact in dynamics is more firmly established upon an experimental basis than the proposition that the effectiveness of a force in producing angular acceleration about any fixed axis in a rigid body is directly proportional to the shortest distance between the direction of the force and the direction of the axis. This truth is well illustrated in the use of the steelyard, or of the rider on the chemical balance, where the divisions on the beam are equidistant.

Having given any axis of rotation A, it is clear that any force F may be resolved into two components, one parallel to A, and the other at right angles to A. The component which is parallel to A can evidently produce no rotation about A, as an axis. A carriage wheel cannot be set in rotation by a push along the axis of the wheel. So far as rotation is concerned, we may, therefore, neglect every component of the force except the one which lies in a plane normal to the axis of rotation.

Now it has been abundantly proved by experience that the turning effect of any force depends only upon these two factors; namely, the component of the force which is at right angles to the axis, and the perpendicular distance from the direction of this component to the direction of the axis.

Definition of Moment of Force. Torque.

81. Accordingly, the moment of any force, R, in a plane with respect to any axis, A, perpendicular to this plane, is defined as the product of the force by the perpendicular distance from the axis to the force. The word " moment " is here employed in precisely the same sense as that used by Shakespeare when he speaks of " enterprises of great pith and moment,"—i.e., in the original Latin sense. Here the *moment* of force measures its *importance* in producing rotation. Torque is used as an exact synonym for moment of force. The former is a rather better term, being shorter, and indicating equally well with the latter its meaning by its etymology. Torque is evidently a special case of " moment " defined in § 67. Hence, if we denote the moment of force by $\bar{L}$, it may be written in any one of the three following equivalent ways. See Fig. 32.

$$L = R \, . \, AP = R \, . \, r \sin \theta = [\bar{r} \, \bar{R}]. \qquad \text{Eq. 77.}$$

From this it will be seen

(1) That the moment of any force vanishes whenever the direction of the force is parallel to the direction of the axis; for then the component R in the plane normal to the axis becomes zero.

(2) That the moment of any force vanishes whenever the direction of the force intersects the direction of the axis; for then the arm r of the moment becomes zero.

(3) That instead of resolving the vector r (Fig. 32), along the perpendicular from the axis to the force, we may

resolve the force along a direction perpendicular to r and to the axis. In this latter case the moment is the product of the distance AM and the component $R \sin \theta$.

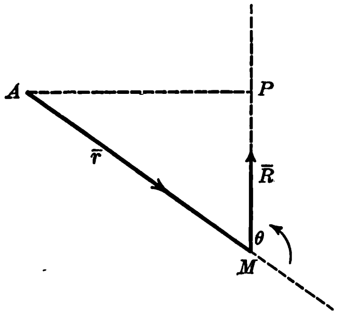


Fig. 32.

(4) That the sum of the moments about any axis of any number of forces acting at any point M is equal to the moment of their resultant about the same axis.

(5) If the rectangular components of the force R be X, Y, Z; if A (in Fig. 32) be chosen as origin; if the components of r (i.e., the coördinates of M) be x, y, z; and if we denote the components of the torque about the axes of X, Y, and Z, by L, M, N, respectively, — then by Eq. 31, § 37, we have

$$\left.\begin{aligned} L &= yZ - zY \\ M &= zX - xZ \\ N &= xY - yX \end{aligned}\right\} \text{Cross-over equations connecting components of Force and Torque.} \qquad \text{Eq. 78.}$$

Note that the three equations in 78 are summarized in the single equation 77.

Definition of Couple.

82. A torque which is exerted upon any body by two equal and parallel, but oppositely directed, forces, is called a *couple*. From this definition it follows that $\Sigma \bar{F} = 0$, and, hence, that the resultant force of any couple is zero. A couple is, therefore, a special case of a torque, its moment of force being measured by the product of either force by the perpendicular distance between them.

RIGOROUS DEFINITION OF ROTATIONAL INERTIA.

83. After this digression upon moments of force, in which we have learned something of their properties and of their measurement, we return to the general equations of motion and proceed to interpret Eq. 76 in which we assumed, by analogy, that the torque required to rotate the particle P about an axis AB (Fig. 31) with an angular acceleration $\frac{d^2\theta}{dt^2}$ is $I\frac{d^2\theta}{dt^2}$.

Let us choose the axis of rotation for the axis of X, then by the first of equations 78, we have

$$yZ - zY = I\frac{d^2\theta}{dt^2};$$

or, remembering Eq. 53,

$$m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) = IA = I\frac{a}{r},$$

where A is the angular and a the linear acceleration along the path. Since the motion of the particle is at all times symmetrical with respect to the X-axis, we may select, for the evaluation of I, the instant at which the particle passes through the XY-plane. Then $z = 0$ and $y = r$ where r is the radius of the circle in which the particle is revolving about the axis of X. Accordingly, the equation becomes

$$mr\frac{d^2z}{dt^2} = I\frac{a}{r}.$$

But at this point $\frac{d^2z}{dt^2}$ is the total acceleration along the path, and is hence exactly equivalent to a. And since, therefore, these two accelerations cancel out, we have for the moment of inertia of a particle,

$$I = mr^2, \qquad \text{Eq. 79.}$$

an expression which is identical with that already obtained (Eq. 68, § 66) by equating angular momentum to moment of linear momentum.

The same result might have been obtained at once by transforming $m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right)$ to polar coördinates, thus giving $mr^2\frac{d^2\theta}{dt^2} = IA$; and hence Eq. 79.

84. Before completing the derivation of the general equation of a rigid body (§ 105), it will be found convenient to first consider the

MOMENT OF INERTIA OF A RIGID BODY.

One is justified by all experience in thinking of a rigid body as made up of particles which are held together by a massless frame, so to speak. And since the mass of any one particle has been proved by numberless experiments to be quite independent of the mass and of the position of its neighbors, it follows that the rotational inertia of a rigid body with respect to any given axis is the sum of the rotational inertias of the respective particles of which the body is composed.

Let us denote by dI the rotational inertia, and by dM the mass of any one particle whose coördinates are x, y, z, and by I the rotational inertia of the entire body. Then by Eq. 79,

$$dI = r^2\,dM;$$

and by definition of ρ, the volume density,

$$dM = \rho\,dx\,dy\,dz.$$

Hence,

$$I = \int_0^M r^2\,dM = \int\int\int r^2\rho\,dx\,dy\,dz. \qquad \text{Eq. 80.}$$

MATHEMATICAL DIGRESSION ON THE COMPUTATION OF MOMENTS OF INERTIA.

85. The engineer and the student of mechanics have each many occasions to compute the rotational inertia of a rigid body. The general and fundamental method for doing this is to evaluate the triple integral of Eq. 80. However, it is not always necessary to employ this general method and to perform the actual quadrature; for a number of " helpful " rules have been discovered which furnish us, in the majority of practical cases, a short cut to the desired result.

We shall, therefore, first take up some illustrations of the general method (which is, in a certain sense, for rotational inertia what the ordinary balance is for linear inertia), and then proceed to derive these more convenient "rules." The ordinary balance gives M by direct observation; the integration indicated gives I by computation.

It is very important to observe (1) that I has no meaning except with reference to a particular straight line — or axis. (2) that the geometry of the rigid body must be definitely stated so that dM may be expressed as a function of r, which is the distance of dM from the axis just referred to.

We now proceed to consider:

(I) A few important special cases in which the rotational inertia is computed by use of the general integral.

(II) A few general theorems concerning moments of inertia.

(III) A valuable rule devised by Routh for computing, without integration, a number of symmetrical cases; and finally

(IV) A few methods for the determination of rotational inertia by experiment.

(I) Some Important Special Cases.

86. *Case* 1. *A uniform fine straight wire.* Find its moment of inertia about an axis perpendicular to its length, and passing through its middle point.

Let L = total length of wire.
λ = mass of unit length = linear density.
dr = element of length.

Then $dM = \lambda\, dr$.
λL = total mass = M;

and one obtains, Eq. 80,

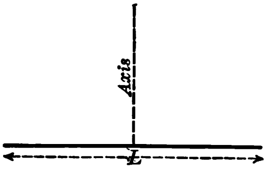


Fig. 34.

$$I = \int_0^M r^2\, dM = 2\lambda \int_0^{\frac{L}{2}} r^2 dr = 2\lambda \Big|_0^{\frac{L}{2}} \frac{r^3}{3} = \frac{\lambda L^3}{12} = \frac{ML^2}{12}. \quad \text{Eq. 81.}$$

The student will find it an interesting exercise to prove that, when the axis of rotation is shifted parallel to itself, so as to pass through one end of the wire, the expression for the rotational inertia becomes

$$I = \frac{ML^2}{3}. \quad \text{Eq. 82.}$$

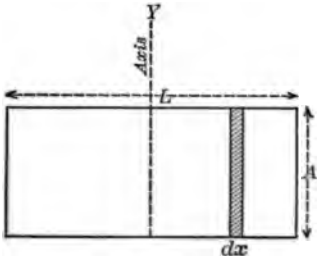


Fig. 35.

87. *Case* 2. *A uniform rectangular plane lamina.* Find the moment of inertia about a central axis lying in its plane and perpendicular to one side, say L.

Let us here use rectangular coördinates, such that the axis of Y coincides with the axis of rotation. Then we may choose for our element of mass a narrow strip parallel to the axis of Y, and having a width dx. Accordingly,

$$dM = \sigma y \cdot dx = \sigma A \cdot dx,$$

where σ = the surface density of the lamina, — i.e., the mass of unit area of the rectangle, — and A is the length of side parallel to the Y-axis.

Since $x = r$ = distance of dM from the axis, our integral becomes

$$I = \int_0^M r^2\,dM = 2\int_0^{\frac{L}{2}} x^2 \cdot \sigma \cdot A \cdot dx = 2\,\sigma A \Big|_0^{\frac{L}{2}} \frac{x^3}{3} = \frac{\sigma A L^3}{12}.$$

But since

$$M = \sigma A L,$$

$$I = \frac{ML^2}{12}, \qquad \text{Eq. 83.}$$

a result which might have been anticipated from the fact that a plane lamina of this kind may be considered as built up of exactly similar elements of length L and width dy, each of which would fall under Case 1.

It is worth noticing that the rotational inertia of such a lamina is a function of its depth A, only in so far as M depends upon A.

88. *Case* 3. *A circular ring of uniform wire.* Find the moment of inertia about a central axis perpendicular to the plane of the circle.

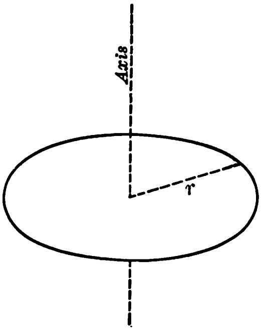


Fig. 36.

Here r, the radius of the ring, is a constant for each element of mass. Accordingly, our integral becomes

$$I = \int_0^M r^2\,dM = r^2 \int_0^M dM = Mr^2. \qquad \text{Eq. 84.}$$

It is evident that the same form of expression would hold for a thin hollow cylinder, about its axis of figure.

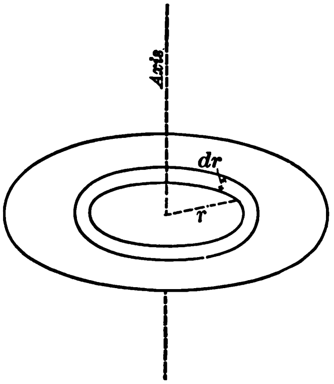


Fig. 37.

89. *Case* 4. *A uniform circular plate.* Find the moment of inertia about a central axis, perpendicular to its plane. Let R denote the radius, and σ the surface density of the plate. Now the general rule is, in any of these cases, to choose a differential element of mass as large as possible, yet satisfying the condition that all parts of the element must be equally distant from the axis. We select, therefore, a circular element of radius r and of an infinitesimal width dr. The area of this element is $2\pi r\ dr$. Hence,

$$dM = \sigma \,.\, 2\,\pi r \,.\, dr.$$

Accordingly, we have for the integral,

$$I = \int_0^M r^2 dM = 2\,\pi\sigma \int_0^R r^3\, dr$$

$$= 2\,\pi\sigma \Big|_0^R \frac{r^4}{4} = \frac{\pi\sigma R^4}{2} = \frac{MR^2}{2}. \qquad \text{Eq. 85.}$$

The student will find it an interesting exercise to prove that in the case of a plane circular annulus whose external radius is R, and whose internal radius is r, the rotational inertia about a central axis perpendicular to the plane is

$$I = M\frac{R^2 + r^2}{2}. \qquad \text{Eq. 86.}$$

90. *Case* 5. *A solid circular cylinder of uniform density* about its geometric axis.

It is at once evident that a cylinder may be thought of as built up of an infinite number of similar disks of infinitesimal thickness, such as that discussed in the immediately preceding case; hence the same form of expression holds, and we have

$$I = M\frac{R^2}{2}, \qquad \text{Eq. 87.}$$

where M is now the mass and R the radius of the cylinder.

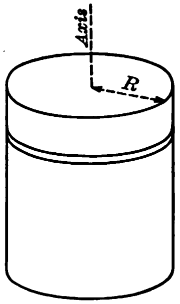


Fig. 38.

91. *Case* 6. *A homogeneous sphere* about any diameter as axis.

Let ρ be the volume density and R the radius of the sphere. Let us imagine the sphere as built up of solid zones, or circular slices, perpendicular to the diameter about which the moment of inertia is sought. Since, from Case 4, we know the rotational inertia for each of these slices, dI, we have merely to add up the rotational inertia of the slices, and thus obtain the rotational inertia of the entire sphere.

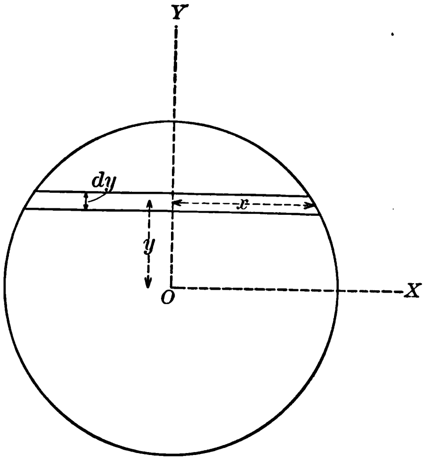


Fig. 39.

Let y denote distances from the center along the axis of rotation, and x denote the radius of any particular slice: then the moment of inertia of each slice is, Eq. 85,

$$dI = dM\frac{x^3}{2}, \quad \text{where } dM = \rho\pi x^2 \cdot dy.$$

Accordingly, our integral becomes

$$I = \int_0^M \frac{x^2}{2}\,dM = \pi\rho\int_{-R}^{+R}\frac{x^4}{2}\,dy.$$

It remains only to express x in terms of y, and then integrate. Since any section of the sphere taken through the center is a circle of radius R, we have, as the equation of this circle,

$$x^2 = R^2 - y^2,$$

and hence,

$$I = \frac{\pi\rho}{2}\int_{-R}^{+R}(R^2 - y^2)^2\,dy = \frac{\pi\rho}{2}\int_{-R}^{+R}(R^4 - 2\,R^2y^2 + y^4)\,dy$$

$$= \frac{\pi\rho}{2}\Big|_{-R}^{+R}\left(R^4y - \frac{2}{3}R^2y^3 + \frac{1}{5}y^5\right) = \frac{\pi\rho}{2}\left(2\,R^5 - \frac{4}{3}R^5 + \frac{2}{5}R^5\right)$$

$$= \frac{\pi\rho}{2}\cdot\frac{16}{15}R^5.$$

Or, since $M = \frac{4}{3}\pi\rho R^3$, one has

$$I = \frac{2}{5}MR^2. \qquad \text{Eq. 88.}$$

92. *Case* 7. *A plane right-angled triangle* about one of the sides containing the right angle. Let a and b denote the lengths of the respective sides of the triangle which include the right angle. Choose the axis of Y so that it coincides with b; the axis of X will then be parallel to the side a.

From Case 1 (Eq. 82) we may write for the moment of inertia of an elementary strip of length x and mass dM, the following expression:

$$dI = \frac{x^2}{3} \cdot dM = \frac{x^2}{3} \cdot \sigma \cdot x \cdot dy.$$

In order to integrate this expression, we have only to express x as a function of y, so as to have but one variable under the integral sign.

Since the hypothenuse of the triangle is a straight line passing through the origin O, we have

$$x = \left(\frac{a}{b}\right) y,$$

which gives us the horizontal width of the triangle as a function of the vertical distance from the apex.

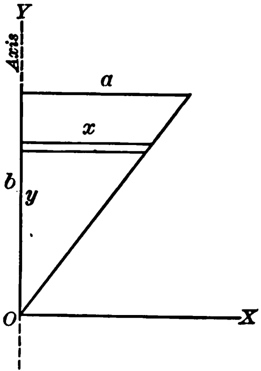


Fig. 40.

Substituting this value of x, and performing the integration, one obtains

$$I = \int_0^b \sigma \frac{a^3}{3\,b^3} y^3\,dy = \frac{\sigma a^3}{3\,b^3} \Big|_0^b \frac{y^4}{4} = \frac{\sigma a^3 b^4}{12\,b^3}.$$

And since $M = \frac{\sigma ab}{2}$, one obtains

$$I = M\frac{a^2}{6}. \qquad \text{Eq. 89.}$$

An interesting variation on the preceding problem is to demonstrate that in the case of an isosceles triangle of base a, the moment of inertia about an axis perpendicular to the base, and passing through the opposite vertex, is

$$I = M\frac{a^2}{24}. \qquad \text{Eq. 90.}$$

Case 8. *An elliptical lamina* about the minor axis.

93. Let us choose as an element of mass a strip of the ellipse parallel to the minor axis which we shall call the axis of Y. Then,

$$dM = 2\,\sigma y \,.\, dx,$$

and

$$dI = x^2 \,.\, 2\,\sigma y\, dx,$$

whence

$$I = 2\int_0^a 2\,\sigma x^2 y \,.\, dx,$$

where a is the length of the semi-major axis. It remains only to express y in terms of x and to integrate. The equation of the ellipse gives this connection, namely,

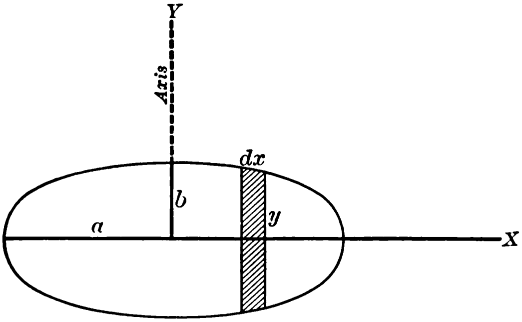


Fig. 41.

$$y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Hence,

$$I = 4.\sigma\frac{b}{a}\int_0^a x^2\sqrt{a^2 - x^2}\, dx.$$

In order to simplify this expression, let us change the variable from x to ϕ by means of the following equation, $x = a \sin \phi$; then,

$$dx = a \cos \phi \, d\phi,$$

and we have

$$I = 4\,\sigma \frac{b}{a} \cdot \; a^4 \int^{\frac{\pi}{2}} \sin^2 \phi \cos^2 \phi \, d\phi = 4\,\sigma b a^3 \cdot \frac{\pi}{16} \; .$$

And since $M = \pi\sigma ab$, in the case of an ellipse, we have

$$I = \frac{Ma^2}{4} . \quad . \qquad \text{Eq. 91.}$$

Had the major axis of the ellipse been chosen as the axis of rotation, it is evident that one would have obtained

$$I = \frac{Mb^2}{4} . \qquad \text{Eq. 92.}$$

For the integration involved here, consult any integral calculus or table of integrals.

(II) General Theorems Concerning Moments of Inertia.

94. In the computation of Moments of Inertia much economy of brain labor and time is secured by use of the following general principles.

THEOREM I. LAMINA. AXIS PERPENDICULAR TO PLANE.

The moment of inertia of a lamina of any form about an axis perpendicular to its plane is obtained by taking the sum of the moments of inertia about any two rectangular axes which lie in the plane of the lamina and which intersect one another in the same point at which the axis of rotation passes through the lamina.

In order to prove this, select the plane of the lamina as the XY-plane, and let O be the point where the axis of rotation passes through the lamina. Call the axis of rotation Z. Then,

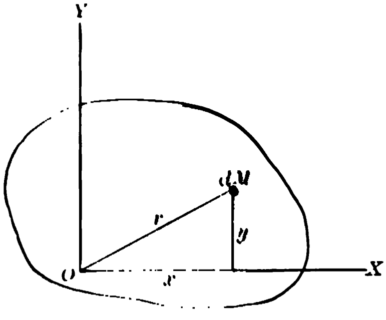


Fig. 42.

$$I = \int_0^M r^2\,dM,$$

where the integration extends over all the particles dM which go to make up the lamina. But for each particle,

$$r^2 = x^2 + y^2,$$

and hence,

$$I_z = \int_0^M (x^2 + y^2)\,dM = \int_0^M x^2\,dM + \int_0^M y^2\,dM = I_x + I_y. \qquad \text{Eq. 93.}$$

Prove by aid of this theorem:

1. That the moment of inertia of a circular disk about any diameter as axis of rotation is $\dfrac{MR^2}{4}$.

2. Find the moment of inertia of an elliptical disk about a central axis perpendicular to the plane of the disk.

3. A rectangular lamina has edges whose lengths are A and B. Find its moment of inertia about a central axis perpendicular to the lamina.

Theorem II. Parallel Axes. Steiner's Theorem.

95. *The moment of inertia of any body about any axis whatever is equal to its moment of inertia about a parallel axis through the center of mass plus the product of the mass of the body by the square of the distance between these two axes.*

Let I denote the required moment of inertia; I_m the moment of inertia about the center of mass. Then, if R is the distance from the center of mass to the axis of rotation, the theorem may be expressed algebraically as follows:

$$I = I_m + MR^2. \qquad \text{Eq. 94.}$$

The lamina represented in Fig. 43 is a section of the body passing through the center of mass, O, and is perpendicular to the axis of rotation which passes through the point A.

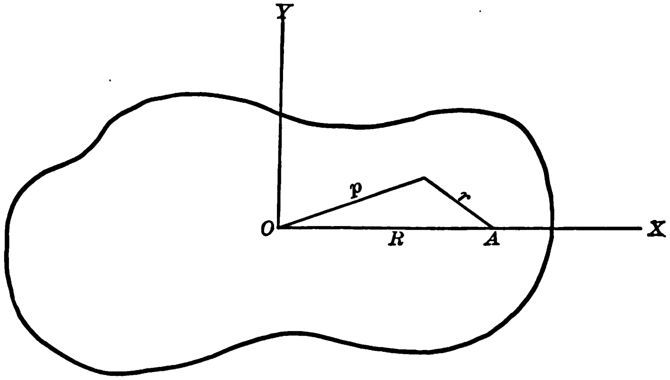


Fig. 43.

We may now consider the entire body as made up of elementary filaments of matter parallel to the axis of rotation. Let dM be the mass of such a filament; let p be its distance from the center of mass, and r its distance from the axis of rotation. Then we have

$$I = \int_0^M r^2\, dM, \text{ and } \quad I_m = \int_0^M p^2\, dM.$$

Now p and R are connected by the following equation,

$$r^2 = p^2 + R^2 - 2\,pR \cos\theta,$$

where

$$\theta = \text{the angle between } p \text{ and } R.$$

Substituting for r in the first integral, we have

$$I = \int_0^M p^2\,dM + R^2\int_0^M dM - 2\,R\int_0^M p\cos\theta\,.\,dM.$$

Let the line OA be chosen as the axis of X, then

$$I = I_m + MR^2 - 2\,R\int_0^M x\,.\,dM.$$

But, as we have already seen (Eq. 60′), the integral part of $x\,dM$ is precisely the quantity which becomes zero when the x's are measured from the center of mass.

Accordingly,

$$I = I_m + MR^2. \qquad \text{Eq. 94.}$$

PROBLEMS.

1. Find the moment of inertia of a thin circular coin about an axis tangent to its edge.

2. Consider all possible axes parallel to any given axis of rotation. About which of these will the moment of inertia be a minimum?

3. Find the moment of inertia of a door about its hinge-line.

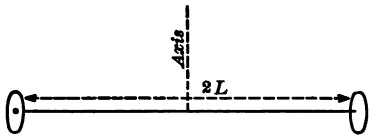


Fig. 44.

4. Two circular disks are carried, one at each end of a massless rod of length 2 L. Suppose the rod to be perpendicular to the plane of each disk at its center, and find the moment of inertia of the system about an axis of rotation which bisects the rod and is perpendicular to it.

THEOREM III. ROTATIONAL INERTIA OF A UNIFORM RIGHT PRISM.

96. *The moment of inertia of a right prism about a central axis perpendicular to the line joining the centers of mass of the two ends is equal to the moment of inertia of the same prism* *considered as a thin line of particles plus the moment of inertia which the prism would have if condensed by endwise contraction into a single thin slice at the axis of rotation.*

Let us divide the prism into an infinite number of thin slices, each of mass dM, and each parallel to the ends of the prism; denote the moment of inertia of any one of these slices with respect to the axis of rotation by dI. Then by Steiner's Theorem, Eq. 94, we have

$$dI = R^2 dM + dI_m,$$

where dI_m is the moment of inertia of the slice about its center of gravity, and R the distance of the slice from the axis of rotation. If now we integrate the right-hand member, we shall obtain the desired moment of the entire prism about the given axis:

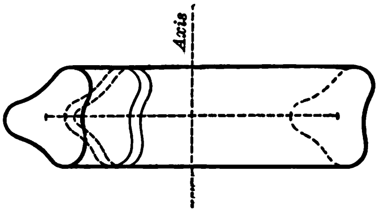


Fig. 45.

$$I = \int_0^M R^2 dM + \int_0^{I_m} dI_m. \qquad \text{Eq. 95.}$$

But the first of these integrals is simply the integral of a heavy line of mass M having the same length as that of the given prism; the second integral is that of a lamina whose mass is M and whose shape is that of a cross-section of the prism. Hence the theorem.

The Six Inertia Constants of a Body.

97. It is necessary to precede the statement and demonstration of the next theorem by a definition of:

Hence

$$I = \int_0^M r^2\, dM$$

$$= \cos^2 \alpha \int_0^M (y^2+z^2)\, dM + \cos^2 \beta \int_0^M (x^2+z^2)\, dM$$

$$+ \cos^2 \gamma \int_0^M (y^2 + x^2)\; dM$$

$$- 2 \cos \beta \cos \gamma \int_0^M yz\, dM - 2 \cos \gamma \cos \alpha \int_0^M zx\, dM$$

$$- 2 \cos \alpha \cos \beta \int_0^M xy\; dM.$$

Employing the definitions of moments and products of inertia which we have already adopted, this becomes

$$I = A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma - 2\,D \cos \beta \cos \gamma$$
$$- 2\,E \cos \gamma \cos \alpha - 2\,F \cos \alpha\, \cos \beta, \qquad \text{Eq. 97.}$$

which is Theorem IV.

Principal Axes.

99. When the three coördinate axes are so chosen that

$$D = E = F = 0, \qquad \text{Eq. 98.}$$

these axes are said to be *principal axes.* In this case,

$$I = A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma. \qquad \text{Eq. 99.}$$

The Inertia Ellipsoid.

100. If now we multiply each side of Eq. 97 by the square of a radius vector r such that $r^2I = 1$, then we have

$$r^2 I = 1$$
$$= Ax^2 + By^2 + Cz^2 - 2\,Dyz - 2\,Ezx - 2\,Fxy, \qquad \text{Eq. 100.}$$

Let OA be the axis of rotation about which the moment of inertia is desired.

Let P be the position of any particle whose mass is dM and whose coördinates are x, y, z. Then the perpendicular distance from P to OA, which we may denote by r, will be the distance of the element of mass from the axis of rotation. In order, then, to find the moment of inertia about the line OA, we have merely to express the integral $\int_0^M r^2 dM$ in terms of the six constants and the direction cosines of the axis OA. This is done as follows:

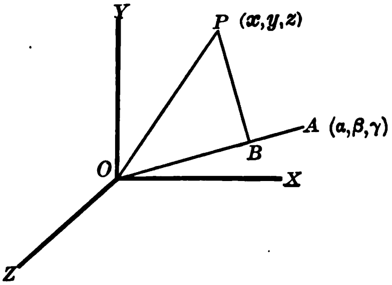


Fig. 46.

$$r^2 = \overline{OP}^2 - \overline{OB}^2.$$

Or, since $\overline{OB}$ is the projection of $\overline{OP}$ upon $\overline{OA}$,

$$r^2 = x^2 + y^2 + z^2 - [OP \cos \widehat{POB}]^2$$
$$= x^2 + y^2 + z^2 - [x \cos \alpha + y \cos \beta + z \cos \gamma]^2.$$

Expanding the square, and remembering that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1,$$

so that
$$x^2 (1 - \cos^2 \alpha) = x^2 (\cos^2 \beta + \cos^2 \gamma),$$

we have

$$r^2 = (y^2 + z^2) \cos^2 \alpha + (z^2 + x^2) \cos^2 \beta + (x^2 + y^2) \cos^2 \gamma,$$
$$- 2\, yz \cos \beta \cos \gamma - 2\, xz \cos \gamma \cos \alpha - 2\, xy \cos \alpha \cos \beta.$$

Radius of Gyration.

102. It is often very convenient to separate the moment of inertia into two factors, one of which is the mass, — a quantity easily determined by the balance, — and the other a geometrical quantity which we may denote by K^2, the quantity K being known as the *radius of gyration.*

It is evident that, if we consider the moment of inertia from this point of view, the defining equation for radius of gyration about any axis is

$$I = MK^2, \qquad \text{Eq. 101.}$$

where M is the mass of the body, and I is the moment of inertia about the same axis to which K refers. Thus the (radius of gyration)^2 of a circular disk, of radius R, about a central axis perpendicular to its plane is $\frac{R^2}{2}$. For a sphere of radius R about any diameter the radius of gyration is $R\sqrt{\frac{2}{5}}$.

As will be evident from Eq. 101, the *radius of gyration* is to be interpreted as the distance from the axis of rotation at which the mass of any given body would have to be condensed in order to produce a fictitious body having the same moment of inertia as the given body.

From the preceding it will be clear that when we once know the radius of gyration of any body, we have merely to weigh the body in order to determine immediately its moment of inertia about the axis for which K is known.

(IV) Experimental Determination of Inertia.

103. For bodies having a simple geometrical form and a uniform density, the moment of inertia is most simply determined by computation from the mass and dimensions of the body. But one has often to deal with bodies of irregular shape and of uncertain density, such, for instance, as a ship's propeller, or a hollow steel magnet, one end of which is fitted with a brass cell and a glass lens.

In such cases the moment of inertia is most readily obtained by experiment, as follows:

The Laboratory Equations.

For small elastic deformations of all kinds, it is well known that the deformation is proportional to the force producing it. This general statement — to a discussion of which we will return later, under the head of Elasticity — is known as *Hooke's Law.* Thus in the case of a vertical spiral spring (Fig. 47) carrying a mass m at its lower end, Hooke's Law would take the following form,

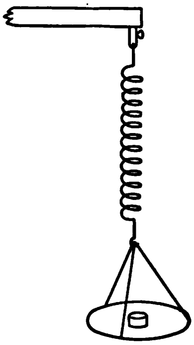


Fig. 47.

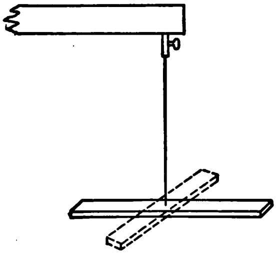


Fig. 48.

$$m \frac{d^2y}{dt^2} = - ky, \qquad \text{Eq. 102.}$$

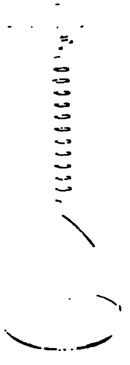
where y is the amount by which the spring is displaced from its position of equilibrium; in short, y is the amount of the deformation, or, as it is often called, the amount of the [illegible] spring. In the same [illegible] and made to rotate [illegible] torque $I\ \frac{d^2\theta}{dt^2}$, then for [illegible]

[illegible] Eq. 103.

[illegible] equations of motion [illegible] Motion, viz. Eq. [illegible] is given by Eq. [illegible]

[illegible] Eq. 48.

[illegible] Eq. 58½.



[illegible] equations, we see that [illegible] oscillation of this type [illegible] the constant of the [illegible] the inertia, m or I, [illegible] loaded. So much is [illegible]

[illegible] we have given a body [illegible] mass m. We have merely [illegible] a scale pan of known mass, m, attached to the lower end of a spiral spring, Fig. 49. Measure its period of vibration T_1, which is also given by the following equation:

$$T_1 = 2\pi\sqrt{\frac{m+w}{k}}.$$

Again measure the period of vibration T_0 of the empty pan m. Then we have

$$T_0 = 2\pi\sqrt{\frac{m}{k}}.$$

From these two equations we may eliminate the constant of the spring, k, and obtain

$$w = m\left(\frac{T_1^2 - T_0^2}{T_0^2}\right), \qquad \text{Eq. 104.}$$

which is a laboratory equation for the determination of mass w in terms of the mass m. It will be observed that we have here neglected the mass of the vibrating spring. Later, § 149, it will be found an interesting problem to find just how much must be added to the mass of the pan to correct for the mass of the spring. It will be found, if we call the mass of the spring S, that $m + \frac{S}{3}$ will represent the effective inertia of the vibrating system composed of the spring and the empty pan. Accordingly the laboratory equation, in its corrected form, will be

$$w = \left(m + \frac{S}{3}\right)\left(\frac{T_1^2 - T_0^2}{T_0^2}\right). \qquad \text{Eq. 105}$$

Analogue.

104. In precisely the same manner we may employ Eq. 58½ to compare moments of inertia.

Perhaps the most convenient form of experiment is that described by Worthington. A circular disk is held on the upper end of a vertical shaft, which rotates easily upon a pivot bearing at the bottom. By a spiral spring of several turns it is held in a definite position of equilibrium, about which it executes angular oscillations. And since it obeys Hooke's Law, these oscillations are very nearly isochronous.

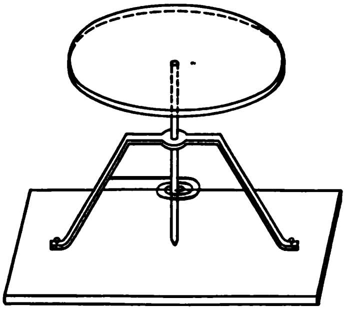


Fig. 50.

Accordingly, we determine the moment of inertia of this disk by allowing it to oscillate first without any load, and second with a load of *known* moment of inertia, say a coaxial circular disk whose dimensions are easily measured. We observe the periods under these two conditions; then calling I and W the moment of inertia of the table and of the load respectively, we have

$$T_0 = 2\pi\sqrt{\frac{I}{K}},$$

and

$$T_1 = 2\pi\sqrt{\frac{I+W}{K}},$$

whence

$$I = W\left(\frac{T_0^2}{T_1^2 - T_0^2}\right) = \left\{\begin{array}{c}\text{Moment of Inertia} \\ \text{of table alone.}\end{array}\right\}$$

Now, knowing the moment of inertia of the table, we can place a load of *unknown* moment upon it, and observing two periods again, solve two similar equations in which W is now an unknown moment of inertia, and obtain

$$W = I\left(\frac{T_1^2 - T_0^2}{T_0^2}\right). \qquad \text{Eq. 106.}$$

Eqs. 105 and 106 thus give us two strictly analogous methods for the experimental determination of masses and moment of inertia.

GENERAL EQUATIONS OF MOTION. CASE OF A RIGID BODY.

105. Returning now from the digression upon Moment of Inertia, entered upon at § 85, we are now in a position to derive and comprehend the six fundamental equations of dynamics, the six equations of which practically all others are special cases, the six equations which are necessary to express in a practical way Newton's Second Law of Motion.

Before passing to the establishment of these equations, — three for translation and three for rotation, — the student should again emphasize to himself the fact that their content and meaning are identical with the content and meaning of Newton's Second Law. If there is any difficulty connected with them, other than that which attaches to Newton's Second Law, it is simply on account of their algebraic form, which is demanded partly by convenience of computation, and partly by the three-dimensional space in which we live.

Translation.

106. If one denotes by X, Y, Z, the components of the resultant of all the external forces acting upon any particle, and if it is agreed, with Newton and D'Alembert, that, in the case of any rigid body, the internal forces just counter-

[illegible] express the transla[illegible] [illegible] summing the [illegible] [illegible] of the body, thus

Eq. [illegible]

[illegible] the summation in the [illegible] [illegible] particle of the body [illegible] [illegible] on the right exten[illegible] [illegible] forces act.

[illegible], we may, in the case

$$[illegible] M[illegible],$$

[illegible] where [illegible] are the [illegible] 60 [illegible]

Eq. 108.

[illegible] When integrated [illegible] the [illegible] the data [illegible] the external [illegible]

Analogue. GENERAL EQUATIONS OF ROTATION.

107. We pass now to the rotational analogue of equations 108. Let us call the moment of inertia of any particle of the body about the axes of X, Y, and Z, respectively, I_x, I_y, and I_z; and, in like manner, angles of rotation about these axes by θ_x, θ_y, and θ_z. Then we may write at once, by analogy,

$$\Sigma I_x \frac{d^2\theta_x}{dt^2} = \Sigma L,$$

$$\Sigma I_y \frac{d^2\theta_y}{dt^2} = \Sigma M, \qquad \text{Eq. 109.}$$

$$\Sigma I_z \frac{d^2\theta_z}{dt^2} = \Sigma N,$$

where L, M, and N are the components about the axes of X, Y, and Z respectively of the external torques acting upon the various particles of the body. Here the summation, in the left-hand member, extends over all particles of the body which have any mass; while the right-hand member is obtained by a summation over all those particles upon which any external force exerts a torque.

But there is one radical and fundamental difference between these equations for rotation and those just obtained (82) for translation, namely, in the case of translation the mass remains a constant wherever the axes of reference be chosen, and the linear inertia of a particle is the same for all three axes; but in the case of rotation, we find that the rotational inertia of any particle varies with the position of the axes of reference, and varies also from one to another of the three axes.

CROSS-OVER EQUATIONS OF MOTION.

108. Accordingly, while the physical idea is best obtained from Eq. 109, it is found that for purposes of computation, the cross-over equations, which translate the values of 109 into terms of linear forces and linear inertia, are far more convenient. This transformation is easily accomplished by means of the cross-over equations for a particle, § 81, namely,

$$\left.\begin{aligned} L &= yZ - zY, \\ M &= zX - Xz, \\ N &= xY - yX. \end{aligned}\right\} \qquad \text{Eq. 78.}$$

Substituting these in 109 we have

$$\left.\begin{aligned} \Sigma I_x \frac{d^2\theta_x}{dt^2} &= \Sigma\,(yZ - zY), \\ \Sigma I_y \frac{d^2\theta_y}{dt^2} &= \Sigma\,(zX - xZ), \\ \Sigma I_z \frac{d^2\theta_z}{dt^2} &= \Sigma\,(xY - yX). \end{aligned}\right\} \qquad \text{Eq. 110.}$$

Another and often convenient form of these equations is obtained by substituting for X, Y, and Z their values obtained from the equations of translations for a particle, § 78,

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= X, \\ m\frac{d^2y}{dt^2} &= Y, \\ m\frac{d^2z}{dt^2} &= Z, \end{aligned}\right\} \qquad \text{Eq. 75.}$$

which gives

$$\left.\begin{aligned} \Sigma\left[m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right)\right] &= \Sigma\,(yZ - zY), \\ \Sigma\left[m\left(z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right)\right] &= \Sigma\,(zX - xZ), \\ \Sigma\left[m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right)\right] &= \Sigma\,(xY - yX. \end{aligned}\right\} \qquad \text{Eq. 111.}$$

The student should again recall to mind the fact that Eqs. 109, 110, and 111 are simply three different methods of expressing the same fact, namely, Newton's Second Law of Motion as applied to any rigid body. The subject of equilibrium will be discussed after the question of energy has been considered.

INDEPENDENCE OF FORCES AND COUPLES.

109. In § 82, a *couple* has been defined as the torque due to two forces which are equal, parallel, and oppositely directed.

It is evident that the resultant force in this case is zero; but the moment of force (or torque) is Fr where F is the value of each of the two forces, and r is the perpendicular distance between their lines of action.

We proceed now to prove that when a couple acts upon a rigid body it does not affect the motion of the center of mass; and conversely, when a given force acts upon a rigid body, it produces the same effect upon the center of mass whether the force produces rotation of the body or not. But, before considering the rigid demonstration of this theorem, it is well for the student to get a clear grasp of its physical meaning by repeating as many as possible of the following series of simple, but excellently chosen, experiments, suggested by Worthington in his *Dynamics of Rotation*, Chapter IX. The first five of these experiments illustrate the fact that there is a point in any free rigid body through which a force may act upon it and produce *translation without rotation.* The last two experiments show that when a couple acts upon a free rigid body, it produces *rotation without translation.*

" *Experiment* 1. Let any convenient rigid body, such as a walking-stick, a hammer, or, say, a straight rod conveniently weighted at one end, be held vertically by one hand and then allowed to fall, and while falling let the observer strike it a smart horizontal blow, and observe whether this causes it to turn, and which way round; it is easy, after a few trials, to find a point at which, if the rod be struck, it will not turn. If struck at any other point, it does turn. The experiment is a partial realization of that just alluded to.

"*Experiment* 2. It is instructive to make the experiment in another way. Let a smooth stone of any shape, resting loosely on smooth, hard ice, be poked with a stick. It will be found easy to poke the stone either so that it shall turn, or so that it shall not turn; and if the direction of the thrusts which move the stone without rotation be noticed, it will be found that the vertical planes containing these directions intersect in a common line. If, now, the stone be turned on its side and the experiments repeated, a second such line can be found intersecting the first. The intersection gives a point through which it will be found that any force must pass which will cause motion without turning.

"*Experiment* 3. With a light object, such as a flat piece of paper or card of any shape, the experiment may be made by laying it, with a very fine thread attached, on the surface of a horizontal mirror dusted over with lycopodium powder to diminish friction, and then tugging at the thread; the image of the thread in the mirror aids in the alignment. The thread is then attached at a different place, and a second line on the paper is obtained.

"*Experiment* 4. Let a rigid body of any shape whatever be allowed to fall freely from rest. It will be observed that, in whatever position the body may have been held, it falls without turning (so long, at any rate, as the disturbing effect of air friction can be neglected). In this case we know that the body is, in every position, acted on by a system of forces (the weights of the respective particles) whose resultant passes through the center of gravity.

"*Experiment* 5. When a body hangs at rest by a string, the direction of the string passes through the center of gravity. If the string be pulled either gradually or with a sudden jerk, the body moves upward with a corresponding acceleration, but again without turning. This is a very accurate proof of the coincidence of the two points.

"*Experiment* 6. A magnet NS lies horizontally on a square-cut block of wood, being suitably counterpoised by weights of brass or lead, so that the wood can float, as shown, in a large vessel of still water. The whole is turned so that the magnet lies in the magnetic east and west, and then released, when it will be observed that the center of gravity G remains* vertically under a fixed point P as the whole turns about it. It is assumed here that the magnet is affected by a horizontal couple due to the earth's action.

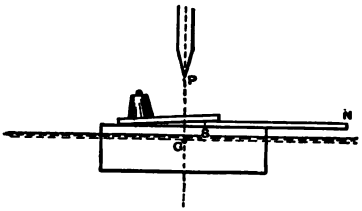


Fig. 50¼.

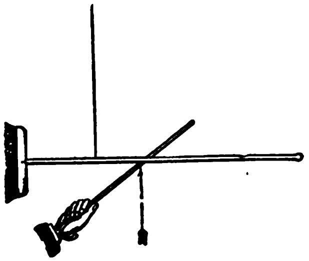


Fig. 50¾.

---

* The center of gravity must, for hydrostatic reasons, be situated in the same vertical line as the center of figure of the submerged part of the block.

WORK AND ENERGY.

110. Having now considered the first three of the five fundamental ideas of dynamics, namely, (1) *inertia* (linear and rotational), (2) *momentum* (linear and angular), and (3) *force and torque,* we pass now to the fourth fundamental conception, that of *energy,* and work.

Work is a term which is used in every-day language with a great variety of meanings; but in Mechanics it is employed always with one signification, namely, the scalar product of a force multiplied by the displacement of the point at which the force is applied.

If, therefore, the linear force, F, is constant and is always directed along the path s, the work W done by the force, will be given by the following simple expression,

$$W = Fs.$$

If, however, the force varies from point to point along the path, and if XYZ are the components of the force, the defining equation for work will be (Eq. 26, § 34)

$$\begin{aligned} W &= \int_0 \bar{F}\, d\bar{s} \\ &= \int_0 (X\,dx + Y\,dy + Z\,dz). \qquad \text{Eq. 112.} \end{aligned}$$

This result is sometimes expressed by saying that work done by any force is the line integral of that force taken along the path of the point of application. It is all-important to remember that, in this integral, F and ds are both to be measured, at each instant, in the same direction.

The reasonableness and naturalness of this definition will be patent to any one who considers that it takes twice as much effort to pump 100 gallons of water out of a well as to pump 50 gallons through the same height; or that it takes twice as much energy to lift a mass of water to a height of 60 feet as it does to lift it to a height of 30 feet.

Analogue.

111. In precisely the same manner, one sees that in hoisting an anchor or in moving a house, the work done on the capstan bar is proportional to the torque and to the angle through which the torque is exerted. The rotation of the capstan through 6 radians will lift the anchor twice as high as a rotation through 3 radians. Accordingly the work done upon any rigid body by a torque, L, is measured by the angular integral of the torque; thus,

$$W = \int_0^{\Theta} L\,d\theta. \qquad \text{Eq. 113.}$$

If the torque is uniform for all values of θ, we may write at once,

$$W = L\theta,$$

otherwise we must integrate as indicated in Eq. 113. In order to pass algebraically from Eq. 112 to 113, one has only to recall the two cross-over equations $L = Fr$ and $s = r\theta$.

DEFINITION OF ENERGY.

112. It is observed that bodies in certain positions, and systems in certain conditions, are able to exert forces through certain distances. Bodies which possess this power of doing work are said to possess *energy*. It is, therefore, customary to define *energy* as *the ability to do work.*

A system may possess energy in virtue of its

1. *Position*, e.g. the block of a pile-driver at the top of its fall.

2. *Shape*, e.g. the coiled mainspring of a watch.

3. *Pressure*, e.g. the steam inclosed behind the piston of a steam engine.

4. *Temperature*, e.g. a tank of hot water which might be employed to operate a heat engine.

5. *Speed*, e.g. a locomotive at high speed may be made to lift itself through a considerable height, without the use of any more steam, by running up grade.

6. *Electrical Condition*, e.g. a charged Leyden jar may be made to ring bells, as in Franklin's "Electric chimes," etc.

But all these various circumstances which are mechanical may be grouped under two heads; so that we may say that a body has energy always in virtue either of its *motion* or of its *position*. The energy of a system which is due to the position (i.e., the relative position of the parts) of that system is called *potential energy*. The energy of a system which is due to the motion of its parts is called *kinetic energy*.

DISSIPATION OF ENERGY.

113. It is well known that if a pendulum be released at any given height, say *h*, above its position of rest, that it will not rise to quite the same height on the other side of the position of rest. As the pendulum moves through the air, some of its energy is frittered away in heat.

In every case of this kind we never get out of a machine all of the energy which we put into it. Some portion of the mechanical energy of a system is always wasted when we attempt to use that energy by transforming it or transferring it. As we shall learn later, this unavoidable waste of energy is brought about by friction; a portion of the mechanical energy is transformed into heat energy, where it is of less use to us than before. *This tendency of energy to assume a less and less available form is known as the principle of the Dissipation of Energy.*

CONSERVATION OF ENERGY.

114. One of the most important generalizations of modern times is that made largely through the efforts of Helmholtz, Joule, and Kelvin about the middle of the last century. Their discovery is that *when allowance has been made for unavoidable wastes,* the sum of the kinetic and potential energies of a body or system of bodies never changes unless through some addition or subtraction of energy from external sources. "Waste of energy," it is important to observe, never means destruction of energy. For there is no evidence that the slightest bit of energy has ever been annihilated. When, therefore, we speak of allowing for "unavoidable wastes," we refer to that energy which has been unavoidably changed into heat or into some other form of energy which, for our purpose, is less useful. *The law of the Conservation of Energy expresses the fact that the sum total of the energy in the universe, or in any isolated system, remains the same.*

This energy can change from one form to another; it tends constantly to become less and less available; but, so far as it is known, it never changes in quantity.

MEASURE OF KINETIC ENERGIES.

115. The principle of the Conservation of Energy may be employed to show us how properly to estimate the amount of kinetic energy in any moving body.

[illegible] upon a [illegible] body [illegible] mass m. [illegible] free from friction [illegible] The effect of the force will be [illegible] the center of mass of the body [illegible] following equation

[illegible]

[illegible] constant during an interval of time. [illegible] constant and the mass m. [illegible] distance [illegible] such that

[illegible] the final velocity of the center of mass

[illegible] force exerted through a

[illegible]

[illegible] the mass m. [illegible] been no waste [illegible]

[illegible] conservation of energy we see that [illegible] which we [illegible] measures its [illegible] we denote by [illegible] have

[illegible] Eq. 114.

Analogue

[illegible] cross-over equation between [illegible]

[illegible] Eq. 77.

If, therefore, we wish to find the kinetic energy of rotation of any rigid body, such, for example, as that of a fly-wheel set in motion by a belt-driven pulley, we may proceed as follows:

Let the tangential force exerted by the belt on the rim of the pulley be F. The work done by the belt in any time, t, will be

$$Fs = Fr\theta = Fr \cdot \tfrac{1}{2}\,\omega t,$$

provided the force is constant. Accordingly,

$$Fs = L \cdot \frac{1}{2}\omega t = IA \cdot \frac{1}{2}\omega t = I\,\frac{\omega}{t} \cdot \frac{1}{2}\omega t = \frac{1}{2}I\omega^2,$$

where I is the moment of inertia and ω the angular speed.

Let us denote the kinetic energy of rotation by E_r; then we have

$$E_r = \tfrac{1}{2}\,I\omega^2. \qquad \text{Eq. 115.}$$

A simpler manner of viewing this matter is perhaps to substitute in Eq. 114 the cross-over equation $v = r\omega$. For we may consider each particle dM, in a rotating body, as having a motion of translation along the circumference of a circle whose radius is the distance of the particle from the axis of rotation. Eq. 114 then becomes directly

$$E_t = \int_0^M \frac{1}{2}\,r^2\omega^2\,dM = \frac{1}{2}\,\omega^2 \int_0^M r^2\,dM = E_r.$$

But ω is a constant for all values of r, and hence, if we call $I = \int_0^M r^2\,dM$, we have

$$E_r = \tfrac{1}{2}\,I\omega^2.$$

It will be observed that, from considerations of energy, we thus arrive at the same expression for rotational inertia, as that obtained under the head of Momentum.

MEASURE OF POTENTIAL ENERGY.

116. Again relying upon the Conservation of Energy, we measure the potential energy of a system in any position or configuration by the amount of work required to bring the system from its standard or zero position into the given position.

If the energy is to be stored in the system by a force (X, Y, Z) acting through a distance S, the work done will be given by the following expression:

$$W = \int_0^S (X\,dx + Y\,dy + Z\,dz)$$

$$= \int_0^S \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) ds. \qquad \text{Eq. 116.}$$

If the kinetic energy is small enough to be neglected, and there be no friction, W will represent the potential energy gained by the system.

Analogue.

In like manner, if the energy is stored up in the system by the action of a torque whose components are (L, M, N), the work done will be for infinitely small displacements $d\theta_x$, $d\theta_y$, $d\theta_z$, about the axes of X, Y, and Z respectively.

$$W = \int_0^\phi (L\,d\theta_x + M\,d\theta_y + N\,d\theta_z)$$

$$= \int_0^\phi \left(L\frac{d\theta_x}{d\phi} + M\frac{d\theta_y}{d\phi} + N\frac{d\theta_z}{d\phi}\right) d\phi. \qquad \text{Eq. 117.}$$

Thus the work required to twist a wire through an angle about its geometrical axis, which we may call the axis of X, is obtained by substituting in the above integral,

$$M = N = 0,$$

and, by Hooke's Law,

$$L = K\theta,$$

where K is the torque required to twist the lower end of the wire through one radian. Accordingly,

$$W = \int_0^\theta K\theta \cdot d\theta = \tfrac{1}{2} K\theta^2. \qquad \text{Eq. 118.}$$

If we might neglect the internal friction of the wire when it untwists, this expression would represent the potential energy of the twisted wire. This, of course, is the analogue of the fact that the potential energy stored up in a spring compressed or extended linearly will be $\frac{1}{2}\ ky^2$ where y is the extension and k the force required to produce unit elongation.

DIGRESSION ON EQUILIBRIUM.

117. At this point it becomes necessary to leave the main thread of our discussion for a moment, to consider a certain class of critical conditions which is of great importance in dynamics, and which is most clearly described in terms of potential energy.

If the speed of a body is zero, the body is said to be at rest; but if the acceleration of the body is zero, it is said to be in *static equilibrium*. Thus, a pendulum bob at the end of its swing is at rest, but it is not in equilibrium; the same bob, at the instant of passing through its lowest point, is in motion, but it is also in static equilibrium. Rest is therefore not the criterion of equilibrium.

But equilibrium may be regarded also from the following viewpoint, which is perhaps simpler. Experience shows that the forces of nature may be divided into two groups. In the first group are included all frictional forces, and all other forces, such that when work is done against them, all this work is converted into some unavailable form of energy, such as heat. Forces of this type are said to be *dissipative* since they dissipate the energy spent in overcoming them through any distance. The forces required to drive an Atlantic liner across the ocean are dissipative; the amount of work required to take a steamer from one port to another depends not only upon the positions of the two ports, but upon the route taken by the steamer between these two ports.

In the second group of forces are included all such that, when work is done against them, this work is stored up in the form of potential energy. Thus the work done in lifting a brick from the ground to a wagon is not wasted in friction, but is accumulated in the system consisting of the earth and the brick, and is ready to be used again in doing work of any kind. Forces of this second type are said to be *conservative.* It will be observed that the work done in overcoming conservative forces is independent of the path. Along whatever route a brick may be taken in lifting it from the ground to the wagon, the same amount of work will be done upon it against the force of gravity. And, from this, it follows that if a body or system of bodies be moved from any given configuration to any other, and if it be then returned to its original configuration, the total work spent will be zero, provided all the forces which are encountered are conservative. Another way of putting this is to say that, *when only conservative forces are involved,* the potential energy of a system depends entirely upon its configuration; so that, for any definite configuration, the potential energy of the system is a definite quantity.

Thus in Eq. 116, if the forces, X, Y, Z, are conservative, the value of W depends entirely upon the limits of the integral, and not at all upon the path of integration. The mathematical condition, therefore, that the forces X, Y, Z, i.e. the forces which resist any change in the configuration of the system, should be conservative, is that

$$\frac{dX}{dy} = \frac{dY}{dx}; \quad \frac{dY}{dz} = \frac{dZ}{dy}; \quad \frac{dZ}{dx} = \frac{dX}{dz}. \qquad \text{Eq. 119.}$$

For if this be true, it follows that X, Y, Z, can each be derived from some function W such that

$$X = -\frac{dW}{dx}, \quad Y = -\frac{dW}{dy}, \quad Z = -\frac{dW}{dz}. \qquad \text{Eq. 120.}$$

In other words

$$Xdx + Vdy + Zdz$$

is, under these conditions, a complete differential of W.

This function W is evidently the integral of Eq. 116, and is the potential energy added to the system in carrying it from its standard configuration to any particular configuration S. The meaning of the minus sign here is simply that a positive amount of work dW must be spent to overcome through a distance dx a force X measured in a sense opposite to dx; in other words, the minus sign results from the opposition of sign between X and dx. For instance, in lifting a weight from the earth, the force Y (i.e., mg) is negative, the displacement dy is positive; hence the product $-Y\,dy$ is a positive quantity, namely, the increment in potential energy.

Equilibrium in the Case of Conservative Forces.

118. It is a matter of common observation that a system may be in equilibrium in several different ways.

A pencil which is balanced on its point enjoys a very precarious stability; while one which is lying upon its side, if it be a circular cylinder, will always be in equilibrium on a level table; if, however, the pencil be one of hexagonal cross-section, it will lie only on one of its flat sides. These three cases typify three kinds of static equilibrium which are met in nature, and which are called *unstable, neutral,* and *stable* respectively.

These three cases are reduced to one general principle as follows. In the law of the conservation of energy, there is nothing which tells us in what direction any change in the system will take place; this law merely describes the fact that in whichever direction the change occurs, the energy of the entire system will remain constant. Thus, so far as the conservation of energy is concerned, a clock might conceivably just as well wind itself up by extracting heat from its surroundings, as to run down and give out heat to its surroundings; just as a globule of mercury will lift itself up (from a flattened shape) against gravity and at the expense of its surface energy.

But from wide and numerous observations the following general principle has been established; namely, every system starting from rest moves in such a way as to diminish its potential energy; or, more briefly, *the potential energy of every system at rest tends to a minimum.* Thus the flattened drop of mercury rises and lifts its center of gravity because, by thus diminishing its surface, its potential energy [consisting of its surface energy and its gravitational energy] assumes a minimum. For the same reason a bent spring tends to straighten itself when the load is removed.

It is further evident that if a system when free to move does not do so, it is because its potential energy is *already* a minimum; i.e., it is in equilibrium.

Bearing in mind this general principle, and also Eq. 120, we see that the general condition of equilibrium is

$$\frac{dW}{dx} = \frac{dW}{dy} = \frac{dW}{dz} = 0,$$

which is precisely the condition that the potential energy of the system, W, should be either a minimum or a maximum. But which of the three kinds of equilibrium does this condition give? A definite answer to this question is found as follows: Let W_0 be the energy of the system in any position of equilibrium. Now suppose the system to have a single degree of freedom, say along the coördinate s, and imagine the system to have received a small displacement ds along this direction.

Since the potential energy W is a continuous function of the single variable s, we may expand it by Taylor's theorem as follows:

$$W = W_0 + \left(\frac{\partial W}{\partial s}\right)_0 ds + \frac{1}{2\,!}\left(\frac{\partial^2 W}{\partial s^2}\right)_0 ds^2 + \frac{1}{3\,!}\left(\frac{\partial^3 W}{\partial s^3}\right)_0 ds^3 + \text{etc.}$$

The change in potential energy brought about in the system by the displacement ds is $W - W_0$; and since $\frac{dW}{ds} = 0$ in virtue of the equilibrium, and since terms higher than the second order may be neglected in comparison with those of the second order, it follows that the sign of $W - W_0$ is determined by the equation

$$W - W_0 = \tfrac{1}{2}\left(\frac{\partial^2 W}{\partial s^2}\right)_0 (ds)^2. \qquad \text{Eq. 122.}$$

First let us suppose that $\left(\frac{\partial^2 W}{\partial s^2}\right)_0$ is *positive.* Then $W - W_0$ is positive; and hence, by the general tendency of potential energy to seek a minimum, the system will slip back into the configuration of equilibrium, W_0. In other words, the *equilibrium is stable when* $\left(\frac{\partial^2 W}{\partial s^2}\right)_0$ *is positive.*

Next let us suppose that $\left(\frac{\partial^2 W}{\partial s^2}\right)_0$ is *negative.* Then $W - W_0$ must be negative; and hence the greater ds is, the less does the potential energy W become; accordingly, the system will, of its own accord, move still farther away from the condition of equilibrium; in other words, the *equilibrium is unstable when* $\left(\frac{\partial^2 W}{\partial s^2}\right)_0$ *is negative.*

But thirdly, it sometimes happens that $\left(\frac{\partial^2 W}{\partial s^2}\right)_0$ and all higher coefficients are zero; in which case W does not vary with s, and the equilibrium is therefore *neutral.*

This criterion applies equally well to rotational and translational equilibrium; the coördinate s being in the former case angular, in the latter linear.

When gravitational forces alone are acting, the general condition of equilibrium may be put in the following form: *All bodies, acted upon by gravitational forces only, tend to assume a position in which the height of the center of gravity is a minimum.*

For most purposes in engineering, the graphical method is a more convenient way of expressing the conditions of equilibrium. Fortunately, most of the motions involved in engineering operations are uniplanar; and therefore the engineer has, for any given link in his mechanism, merely to plot the forces acting upon it, thus forming the ordinary vector polygon. If this polygon be closed, the link is in equilibrium, so far as translation is concerned; if the polygon is not closed, the vector which is required to close it will give the force required to hold the link in equilibrium. This subject is pursued farther in the beautiful science of *Graphical Statics.*

Analogue.

In like manner, the vector sum of the torques must add up to zero; i.e., the sum of the moments of forces must be zero about any axis whatever in order that the link may be in rotational equilibrum.

The question of equilibrium under dissipative forces will be discussed under the head of *Friction.* Equilibrium in fluid and elastic bodies will be treated under the heads of Hydrostatics and Elasticity respectively.

EXAMPLES FOR PRACTICE.

1. A boy rolls a hoop on the sidewalk. Find the ratio of its energy of rotation to its energy of translation.

2. A butcher's spring balance is elongated 1 inch by a force of 3 pounds. How many complete oscillations per second will it make when loaded with a mass of 5 pounds?

3. An electric telegraph cable 1 square inch in cross-section and 80 feet long hangs from the stern of a vessel. If the cable weighs 2 pounds per linear foot, how much work will a hoisting engine perform while winding in the cable?

4. What work per second will be required of an electric motor at the mouth of a mine 2000 feet deep to hoist a bucket containing $\frac{1}{2}$ ton of ore in 30 seconds?

5. In the preceding problem, how long before the bucket reaches the top of the mine should the current be cut off the motor in order that the bucket may reach the top of the mine with zero speed?

6. Three springs are stretched by equal forces. Their extensions are in the proportion of 5 : 10 : 20. Find the ratio of the work done on the respective springs. *Shearer No.* 318.

7. A hollow and a solid sphere roll down an inclined plane at the same time and from the same height. Which will reach the bottom first? No slipping. Radii equal. *Shearer No.* 381.

8. A uniform beam 20 feet long weighs 400 pounds and is supported at each end. Weights of 100, 200, and 300 pounds are placed on the beam at distances of 6, 8, and 10 feet respectively from the left-hand end. Find how the total weight is divided between the two supports.

9. An ordinary lever safety-valve on a boiler has a diameter of 3 inches; the distance from the center of the valve to the fulcrum of the lever is 2 inches. The lever is a uniform bar 32 inches long and weighing 4 pounds. The sliding ball weighs 40 pounds. Find how far from the fulcrum this ball must be placed in order that the steam may blow off at 80 pounds per square inch.

10. Find the radius of gyration of a sphere rolling on a table, about a line on the table passing through the point of contact.

11. Two sides of a triangle represent in direction and amount two forces, taken in order. Show that the sum of the moments of these two forces is the same with respect to every point in a line parallel to the third side of the triangle.

12. In helping a wagon up a hill, why do men generally take hold of a spoke in the rear wheel rather than push on the box of the wagon?

13. A two-wheeled cart together with its load weighs 1800 pounds. The wheels are 4 feet in diameter. Find the horizontal force required to pull the cart over the edge of a 2-inch plank.

POWER.

119. We pass now to the fifth and, for our purpose, last of what may be called the fundamental conceptions of dynamics; namely, *Power*.

This, like force and energy, is a word which is employed in every-day life with considerable variety of meaning; but, in mechanics, *power* is always used to denote the *rate of doing work*. The proper measure of power is therefore the differential coefficient of work with respect to time, and its defining equation is written

$$\text{Power} = P = \frac{dW}{dt}. \qquad \text{Eq. 123.}$$

The fundamental idea involved in power is rapidity of working. The reason it is more fatiguing to ride a wheel up hill than to ride it along a level stretch at the same rate, is that in the former case the rate of working is generally higher. The main difference between a 10-horse-power motor and a 5-horse-power motor is not that the one has done, or is doing, twice as much work as the other, but that the one *can* work twice as rapidly as the other.

As an exact synonym for power, the word " activity " is sometimes used. It was first employed, in this sense, by Newton, to indicate the product of a force by the velocity of its point of application. This usage has been followed by Lord Kelvin and by many others engaged in physical research and engineering. The student should see clearly before going farther that the definition of activity as given by Newton is identical with that of Eq. 123; also that, since power is a limiting ratio between two scalar quantities, it is itself a scalar quantity.

THE UNITS OF KINEMATICS AND DYNAMICS.

120. We have now covered, in a very elementary way, the five fundamental ideas involved in the subject of mechanics; namely, *inertia, momentum, force, energy,* and *power*. We have discussed some of the various conventions and some of the experimental relations which connect these quantities. These relations which have been established are true, whatever consistent system of units be employed for the measurement of the quantities involved.

As a matter of fact, however, the English-speaking world employs only two systems of units. One of these is the so-called Centimeter-gram-second system, generally written "C. G. S. system," which is used in pure science and in electric engineering; the other is the "Engineer's system," used in applied science. We proceed to describe these two systems. The so-called "absolute British system" is a mere curiosity of the text-books on physics.

I. THE C. G. S. SYSTEM. FUNDAMENTAL UNITS.

121. Here we select as our fundamental units the following three, — *length, inertia,* and *time.* The reason for selecting these three, instead of some other three from which all physical units might equally well be derived, lies in the fact that standards of each of these three are easily preserved. They are called "fundamental standards" because they do not hinge upon any other quantities; their values are purely arbitrary, a mere matter of convention.

Standard of Length. Unit of Length.

122. The *standard* of length is the distance between two marks, measured at the proper temperature, on a certain platinum-iridium bar preserved in the International Metric Bureau at Sevres near Paris. This distance is called a *meter.* The *unit of length* usually employed is one hundredth part of this distance, and is called a *centimeter.*

The fact that the meter is so chosen as to be very nearly one ten millionth of the distance from the equator to the pole of the earth, is interesting; but it in no wise enters into the definition of the unit of length.

Unit of Angle. Radian.

123. The unit of angle is independent of all other units, and is defined as follows: If two straight lines lying in the same plane differ in direction they will intersect. About the point of intersection as center, and in the plane of these two lines, draw a circle. These lines are said to include a unit angle when they intercept an arc of the circle equal in length to its radius. This unit is called the *radian*. Degrees, minutes, and seconds are also frequently used in the C. G. S. system, since angular measure is in no wise dependent upon the three fundamental units; and is, indeed, independent of the size of any physical unit.

Standard of Mass. Unit of Inertia.

124. A certain piece of platinum-iridium, also preserved at Sevres, has been agreed upon by the entire scientific world as the *standard* of mass. It is called the *kilogram.* The thousandth part of this standard is employed as the *unit of mass,* and is called the *gram.* Since we employ the terms mass, inertia, and quantity of matter as strict synonyms, it is evident that the gram is our unit of translational inertia.

Unit of Rotational Inertia.

125. The defining equation for moment of inertia shows us that this quantity will be unity when a unit of mass is distributed at unit distance from the axis of rotation.

A standard of rotational inertia might easily be prepared and preserved; but this is unnecessary, since our defining equation, § 84,

$$I = \int_0^M r^2\,dM = Mk^2, \qquad \text{Eq. 80.}$$

connects I with the standards of mass and length. A suitable name for the unit of moment of inertia would be rather convenient.

Standard of Time. Definition of the Second.

126. The mean time occupied by one apparent revolution of the sun about the earth is, so far as is known, one of the most constant quantities met with in nature. It has been chosen, therefore, as our standard of time; and $\frac{1}{86400}$ part of this interval, which is called the *second*, is used by the scientific world as the *unit of time.*

Derived Units.

127. Having now determined upon our fundamental units and upon our definitions of dynamical quantities, we have no choice left as to the proper units to be employed for the remaining quantities. It is a mere matter of interpreting the defining equations.

Units of Speed.

128. Thus, if linear speed is defined (§ 26) as the differential coefficient, $\frac{ds}{dt}$, the *unit of linear speed* is that which, if it remain constant for one second, will carry a point through a distance of one centimeter. No name has been given to this unit.

In like manner, the *unit of angular speed,* $\frac{d\theta}{dt}$, is that rate of rotation which will, if it remains constant, carry a rigid body through an *angle of one radian in one second.* No name has yet been assigned to this unit; though a most excellent English synonym for angular speed in general has been suggested by Professor W. K. Clifford, namely, " spin."

Units of Acceleration.

129. By definition, the linear acceleration *in any direction* is the rate at which the linear speed *in that direction* is changing, Eq. 38. Hence a particle has *unit acceleration* in any direction when its speed in that direction is changing at the rate of *one unit of speed per second;* i.e., at the rate of one centimeter per second per second.

In like manner, a rigid body in rotation about any axis moves under *unit angular acceleration* when it changes its angular speed at the rate of one unit per second; i.e., at the rate of *one radian per second per second.* As yet no name has been assigned to either unit of acceleration. Professor Karl Pearson has suggested the names "spurt" and "shunt" for these two accelerations respectively.

Units of Momentum.

130. Naturally the *unit of linear momentum* (mv) is that of unit mass moving with unit speed; i.e., a mass of one *gram moving at the rate of one centimeter per second.* Likewise a body has *unit angular momentum* ($I\omega$) when its momentum is equivalent to *unit of rotational inertia rotating at the rate of one radian per second.*

Units of Force and Torque. The Dyne.

131. Since the force acting upon any body is the product of the acceleration which it produces in this body multiplied by the mass of the body ($m\,a$), we have no liberty of choice as to a unit of force, Eq. 71.

The C. G. S. unit of force is accordingly *that force which will produce unit acceleration in a mass of one gram.* This unit is called the *Dyne.*

[illegible]

[illegible]

Unit of Energy and Work. The Erg.

[illegible] energy [illegible] is [illegible] energy [illegible] the same [illegible] the unit [illegible] of work [illegible] the same [illegible] in the same [illegible] which is simply [illegible] work. From the [illegible] seen that [illegible] work. [illegible] larger unit of [illegible] which is ten million [illegible]

Unit of Power. The Watt.

133. [illegible] power, $P = \frac{dW}{dt}$.

[illegible] when it does one [illegible]

[illegible] for the reason that [illegible] with the power [illegible] prime movers of an [illegible] generator [illegible] motor [illegible] accordingly, the unit [illegible] practice [illegible] million times the C. G. S. unit, [illegible] the *watt*. In other words, the activity of any agent is *one watt when it performs work at the rate of one joule per second.* When power is produced or absorbed in large quantities, the kilowatt (i.e., 1000 watts) is found to be a still more convenient unit.

The history of the absolute C. G. S. system of units, and especially its later developments beginning with the work of Gauss and Weber in Magnetism and continuing through the labors of the Committee of the British Association for the Advancement of Science on Electrical Standards (1861–1864), is one of the most interesting chapters in physical science.

II. THE ENGINEER'S SYSTEM. FUNDAMENTAL UNITS.

134. We pass now to the consideration of those units which have grown up, so to speak, with the engineering practice of the last three centuries, and which are employed by English-speaking people the world over.

It goes, almost without saying, that such a system is more " natural " than the preceding, — whatever that may mean, — but it does not follow from this that the Engineer's system is better or more logical. Each, indeed, is eminently adapted to its purpose.

The three *fundamental units* which the engineer selects are those of *length, time,* and *force,* in contrast to *length, time,* and *mass,* which are used in the absolute system.

Unit of Length.

135. His standard of length is the yard; his unit of length is one third of the yard, and is called the *foot.* In Great Britain, the yard is, by definition, the distance between two rulings on a certain bronze bar preserved in the Standard

Office at London. In the United States the yard is defined as $\frac{36}{39.37}$ of the standard meter at Sevres. The difference between these two values is so slight as to be perfectly negligible in all engineering operations.

Unit of Angle.

136. The various units of angle employed by engineers are identical with those used in the C. G. S. system, namely, whole revolutions, radians, degrees, minutes, and seconds.

Unit of Time.

137. The second is employed in engineering with precisely the same meaning as in the C. G. S. system, the standard interval of time being the mean solar day.

Unit of Force. The Pound.

138. This unit is clearly described by Rankine (*Civil Engineering*, page 134, sixth ed.) as follows: "The British unit of force is the *standard pound avoirdupois;* which is the weight in the latitude of London, and near the level of the sea, of a certain piece of platinum kept in the Exchequer (now the Standards) Office." This unit is employed by practically all English-speaking engineers.

Derived Units. Speed.

139. As might be expected, a particle is said to have *unit speed* when it moves at the rate of *one foot per second.* Two units of angular speed are widely used; namely, the *radian per second,* and the *revolution per minute.* The symbol for the latter is R. P. M.

Acceleration.

140. Consistently with the preceding, *unit acceleration* is that which changes the speed of a particle at the rate of unit of speed per second; i.e., one foot per second per second. With this goes the corresponding unit of angular acceleration; namely, the radian per second per second.

Unit of Mass. The Slug.

141. The equation, which, in the C. G. S. system, was employed as the defining equation of *force*, is used by engineers to define the unit of *mass;* namely, $m = \frac{F}{a}$. The value of m will be unity when $F = a = 1$. Accordingly, the *unit of mass* is defined as that *mass in which a force of one pound will produce unit acceleration.* Now we know by experiment that, at London, the weight of any mass will produce in that mass an acceleration of 32.2 units. Accordingly, the weight of one pound will produce an acceleration of 32.2 in a mass of one pound. If, therefore, we are to choose our mass of such a size that the force of one pound will produce in it only *unit* acceleration, it is clear that this mass must be 32.2 times as large as the mass of one pound. Accordingly, the engineer's unit of mass is said to be 32.2 pounds. This unit of mass, 32.2 pounds, has come to be rather widely known as the " slug." From this it is clear that the engineer uses the word " pound " sometimes to denote force, and sometimes to denote mass: but he is never ambiguous; for, if the context leaves any doubt as to the meaning, the prefix " force of " or " mass of " will always serve to make the matter clear.

Some people have imagined that an ambiguity might arise with this system of units, from the fact that a given quantity of merchandise, say a thousand pounds of iron, is attracted by the earth with a greater force at some places than at others. Thus a cargo of iron, if measured *in units of force*, would shrink many pounds in going from New York to the equator. But the fact is that the engineer does not measure iron in this manner. On the contrary, he measures the iron at New York by balancing it against a standardized mass of one pound; when he gets to the equator, or, say, Panama, he does the same thing. Consequently the acceleration of gravity is eliminated at each place; so that there is no gain or shrinkage of " weight," in the ordinary engineering sense of that word, in sending merchandise from one part of the globe to another.

The custom of writing " lbs." to denote mass, and " pounds " to denote force, has become quite widespread. Professor DuBois of Yale has added the most excellent suggestion that the pound, when used as a unit of mass, should not only be written " lbs.," but should be pronounced " libra " (plural *libras*), in view both of clearness and etymology.

$$1 \; Slug = 32.2 \; lbs.$$

Unit of Work and Energy. The Foot-Pound.

142. Having once defined work as the scalar product of force and distance, the engineer's unit of work becomes the work done when a force of one pound is exerted through a distance of one foot in the same direction. This unit is known as the *foot-pound.*

It is evident that two quantities of energy measured in this unit at different points of the earth will not be strictly comparable, unless corrected for the difference in gravitational acceleration between the two points. For the work done in lifting a weight is not completely determined when we know the mass of the weight and the vertical distance through which it has been lifted. Thus Rowland found it necessary to take into consideration the difference in gravity between Manchester and Baltimore when he compared his value for the mechanical equivalent of heat with that obtained by Joule.

Unit of Power.

143. The unit of power introduced by James Watt more than a century ago, namely, 550 foot-pounds per second, has been in constant use ever since. This value was his estimate of the rate at which an average horse works, and was therefore named the *horse-power*, and is generally indicated by the symbol " H.P."

It is approximately equivalent to 746 watts. The electrical engineer uses the practical unit of the C. G. S. system, viz., the kilowatt, which is sometimes called the " electrical horse-power."

CHAPTER III.

SOME APPLICATIONS OF GENERAL PRINCIPLES. SPECIAL PROBLEMS.

144. Having now covered, in a brief way, the general principles of dynamics, we proceed to extend the subject through the solution of a number of important problems and exercises which may all be solved as special cases of these general principles.

Problem 1.

145. *Robin's Ballistic Pendulum.* The problem is to find the muzzle speed of a rifle ball.

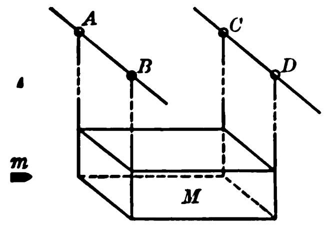


Fig. 51.

A rectangular parallelopiped of wood is suspended by four parallel threads, each of length l, attached one to each of the four upper corners of the wooden block. The effect of this suspension is to make the block vibrate as a simple pendulum of length l, rotating about an axis parallel to the line joining A and B, or C and D (Fig. 51). Let m indicate the mass, and v the speed of the bullet, while M indicates the mass of the pendulum. Let us now consider the system which is made up of the bullet and the pendulum *just before* the bullet strikes the block. The momentum of this system is mv. Suppose the striking of the bullet to impart a speed V to the system, which now consists of the rigid block with the bullet rigidly embedded within it. Since no *external* force has acted upon the system *during collision*, we have, by Newton's First Law, § 70,

$$mv = (M + m)\,V, \qquad \text{Eq. 124.}$$

in which v is the quantity we wish to determine, and in which V is also unknown. In order to determine each of these quantities, we shall need another equation connecting V with measurable quantities. This is obtained from the law of the conservation of energy as follows:

At the instant *immediately after* the bullet strikes, the energy of the system is all kinetic, the pendulum is at its lowest point, and the motion is horizontal. When, a moment later, the pendulum reaches the end of its swing (θ), the energy is all potential, its amount being $(M + m)\,gh$ where $h = l\,(1 - \cos\theta)$, the vertical height through which the system has been lifted.

Equating the total energy in each of these two positions, we have

$$\tfrac{1}{2}\,(M + m)\,V^2 = (M + m)\,g \cdot l\,(1 - \cos\theta). \qquad \text{Eq. 125.}$$

Eliminating V between equations 124 and 125, we have

$$v^2 = 2\left(\frac{M + m}{m}\right)^2 gl\,(1 - \cos\theta),$$

where θ is defined by the following equation,

$$\frac{s}{l} = 2 \sin\frac{\theta}{2},$$

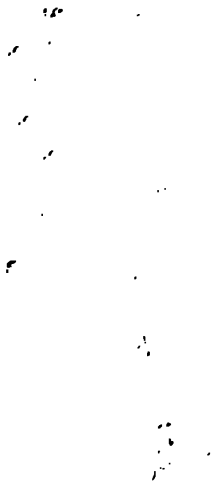
s being the displacement of the block as measured by a fine thread attached to the block and drawn out over the table underneath the pendulum.

[illegible] me

[illegible] them becomes

[illegible]

[illegible] to determine [illegible] between. [illegible] ordinarily [illegible] between [illegible]



[illegible] simple [illegible] small [illegible] strictly [illegible] time [illegible] is [illegible] second [illegible] weight [illegible] equilibrium [illegible] propor- [illegible] angular dis- [illegible] times [illegible] is [illegible] itself; [illegible] enables [illegible] were a

Fig. 67

right body [illegible] means of a spiral spring [illegible] But we have already (§ 104) determined the period of any system performing angular oscillations according to Hooke's Law, namely,

$$T = 2\pi\sqrt{\frac{I}{K}}, \qquad \text{Eq. } 58\tfrac{1}{2}.$$

where K is the torque required to produce unit angular displacement. Worthington calls K the "restoring moment." This name is all right provided it is understood clearly that K is *not* a torque, but the ratio of a torque to an angular displacement. Let us denote by l the length of the simple pendulum, and by m the mass of the bob; then, by definition,

$$K = \frac{mgl \sin\theta}{\theta} = \frac{mgl\theta}{\theta} \left\{\text{approximately for small values of } \theta\right\}$$

and hence,

$$T = 2\pi\sqrt{\frac{I}{K}} = 2\pi\sqrt{\frac{ml^2}{mgl}} = 2\pi\sqrt{\frac{l}{g}}. \qquad \text{Eq. 127.}$$

To find the period of such system in terms of quantities measurable in the laboratory is the essential object of the problem. From the preceding it is evident that the motion of the pendulum bob, through small ranges, is practically "Simple Harmonic Motion," a fact involved in our initial assumption that the system obeys Hooke's Law.

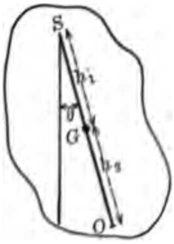


Fig. 53.

Problem 3.

147. *The Physical Pendulum.* As a matter of fact, all actual pendulums are physical pendulums. Let S be the projection of the axis of rotation of any rigid body upon its plane of motion. Let G denote the center gravity of the oscillating body.

If now in Fig. 53 we begin at the point S and lay off from the point, S, a line SO, passing through G and equal to l, the point O, thus determined, is called the *center of oscillation*. This point has several remarkable properties.

(i) It is evident that if the entire mass of the pendulum were concentrated at this point, the period would not be altered; for we would then have a simple pendulum of length l. All points of the physical pendulum which are nearer the axis of rotation than O will actually vibrate more slowly than if they were free; and all points which are farther away from S, than is O, will vibrate more rapidly than if they were free.

(ii) The center of oscillation is always farther away than the center of gravity, G, from the axis S. To prove this, let us denote by I_g the moment of inertia of the pendulum about its center of gravity, and by I_s the moment of inertia about the center of suspension, S. But, by Steiner's Theorem, § 95, we have

$$I_s = I_g + mh_1^2;$$

and hence, by Eq. 129,

$$l = \frac{I_g + mh_1^2}{mh_1} = h_1 + \frac{I_g}{mh_1}. \qquad \text{Eq. 130.}$$

And since the term $\frac{I_g}{mh_1}$ is always positive, it follows that $l > h_1$, which is the property stated above.

(iii) But the most remarkable property of the point O is the fact that it is interchangeable with S; that is, if the pendulum be suspended from O, then S becomes the center of oscillation; in other words, the pendulum has the same period when suspended from O as when suspended from S.

6. A piece of flat steel spring is clamped horizontally in a vise. At the free end of the spring is attached a heavy steel ball, in comparison with which the mass of the spring may be neglected. Suppose the ball to weigh 1 kilogram, and to be depressed through a distance of 5 millimeters by the addition of a weight of 40 grams. Find the period of vibration of the ball in a vertical plane.

7. How many seconds a day would a pendulum clock lose after being carried from Washington, where the acceleration of gravity is 980.1, to Singapore, where this quantity is 978.1?

8. A mass is suspended by a vertical spiral spring, and vibrates along a vertical line with an amplitude of 2 inches. What period must this simple harmonic motion have in order that the suspended mass on reaching its highest point will not exert any pull upon the spring?

9. A U-tube open at both ends is partially filled with liquid. Let a denote the total length of the tube which is filled with liquid, and let g denote the acceleration of gravity. Prove that, when the liquid in one arm of the tube is slightly elevated or depressed (with reference to the liquid in the other arm) and then released, the period of vibration of the liquid will be $2\pi\sqrt{\frac{1}{2}}\cdot\frac{a}{g}\cdot$ Perry, *Applied Mechanics*, page 553.

10. A particle describes a simple harmonic motion with a period of $\frac{2}{5}$ second, and an amplitude of 6 inches. Find the phase, speed, and acceleration of the particle at an instant when its displacement is 3 inches in the positive direction.

Problem 4.

148. *The Reversible Pendulum.* It is evident from Eq. 128 that the length, l, of a simple pendulum having the same period as any given compound pendulum, is given by the following equation, where I_s is the moment of inertia of the given pendulum about the point of suspension:

$$l = \frac{I_s}{mh_1}. \qquad \text{Eq. 129.}$$

of vibration about A as about B. When, however, two asymmetric points such as S and B are found, for which the periods are the same, then we have a truly reversible pendulum, and the distance SB is the length of the equivalent simple pendulum.

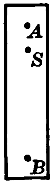


Fig. 55.

The importance of the reversible pendulum lies in the fact that when the centers of oscillation and suspension have once been discovered, we have only to measure the distance between them in order to obtain the length of the equivalent simple pendulum. If in addition we measure the period, we may by Eq. 127 determine the acceleration of gravity with high accuracy.

NOTE. Eqs. 127 and 128 derived for the periods of the simple and compound pendulums are strictly true only for exceedingly small amplitudes, inasmuch as we have assumed $\sin\,\theta = \theta$. The more exact laboratory expression for the period of a pendulum vibrating through an amplitude θ — the proof of which would here lead us too far afield — is

$$T = 2\pi\sqrt{\frac{l}{g}}\left\{1 + \left(\frac{1^2}{2}\right)\sin^2\frac{\theta}{2} + \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\right)\sin^4\frac{\theta}{2}\right.$$
$$\left. + \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)^2\sin^6\frac{\theta}{2} + \text{etc.}\right\} \quad \text{Eq. 133.}$$

An elegant derivation of this equation will be found in Webster's *Dynamics*, pages 45–48.

Problem 5.

149. *To determine the equivalent mass of a loaded vibrating spiral spring:* an illustration of the manner in which the Law of the Conservation of Energy is employed in solving actual problems in dynamics.

If we attach the upper end of a spiral spring to a beam, and then attach a mass m to the lower end of the spring, the oscillations of the mass will be practically simple harmonic in character. If we denote the stiffness of the spring (i.e., force required to produce unit displacement) by k, and *if we neglect the mass of the spring*, the equation of motion will be (Eq. 47, § 49)

$$m \frac{d^2y}{dt^2} = -ky.$$

This is, of course, nothing but an algebraic expression for Hooke's Law applied to this particular spring; and since it satisfies the criterion for S. H. M., we may write at once for the period of vibration of the spring,

$$T = 2\pi\sqrt{\frac{m}{k}}.$$

But, as a matter of fact, the spring never is massless, and in many cases we require a degree of accuracy in computing the period, T, which makes it necessary to take into account the mass of the spring. For of course this spring is vibrating as well as the mass m, and hence the period of any such system is always a little larger than it would be if the spring were entirely devoid of mass; that is to say, the actual period is always a little greater than $2\pi\sqrt{\frac{m}{k}}$.

Accordingly, the problem now is, "To determine how the period of a loaded spiral spring is affected by the inertia of the spring itself."

In the case of any elastic body, oscillating about its position of equilibrium, the energy is entirely kinetic when passing through the position of equilibrium.

The various particles of the spring itself contribute to the effective inertia unequal amounts, since particles near the top of the spring have, at any instant, a smaller acceleration than those near the bottom.

To find the effective inertia of the whole spring, let us calculate the energy of the system consisting of the spring and the mass which it carries. Call M the effective inertia of the system, and m the mass of the attached weights.

Let V be the speed of the mass attached to the spring at the instant of passing its position of equilibrium. The speed at any point on the spring, at this instant, is proportional to its distance from the top, so that at a point y centimeters from the top, the speed is $\frac{Vy}{L}$, where L is the length of the spring measured along the axis.

Let μ be the linear density of the spring. The kinetic energy of the element of length, dy, will then be

$$\frac{1}{2}\,\mu\,dy\cdot\left(\frac{Vy}{L}\right)^2.$$

Hence the kinetic energy of the entire system is

$$K.E. = \frac{1}{2}\,MV^2 = \frac{1}{2}\,mV^2 + \frac{1}{2}\,\frac{\mu V^2}{L^2}\int_0^L y^2\,dy$$

$$= \frac{1}{2}\left(m + \frac{S}{3}\right)V^2$$

where S is the total mass of the spring, so that $M = m + \frac{S}{3}$. Accordingly, to obtain the correct period we must add to the load of the spring one third of the mass of the spring itself. If we let $M = m + \frac{S}{3}$, then we may call M "the equivalent mass" of the load, since a mass, M, attached to a massless spring of the same stiffness as that of the actual spring, will have an equal period.

Problem 6.

150. *To find the angular velocity and the reaction at the bearings of a rigid body, capable of rotation about a fixed axis, when the body is given a sudden blow.*

Imagine a rigid body, such as a door, capable of rotation about a fixed axis, but initially at rest. Let this body be set into rotation by a sudden blow whose impulse ($F\,dt$) is indicated by P. Let (X, Y, Z) be the components, not of the force F, but of the impulse P. Let the blow be delivered to the body at a particle m whose coördinates are (a, b, c), with reference to the point O, as origin. Let the axis of rotation be chosen as the Y-axis, and suppose it to be held in position by two bearings, one at O, and one at H. When the blow is struck, we may imagine the axis (not the body) held in equilibrium by two impulses R_0 and R_h applied at the respective bearings and just equal to the impulsive reactions at the respective bearings. Let the components of these two impulses be (X_0, Y_0, Z_0) and (X_h, Y_h, Z_h).

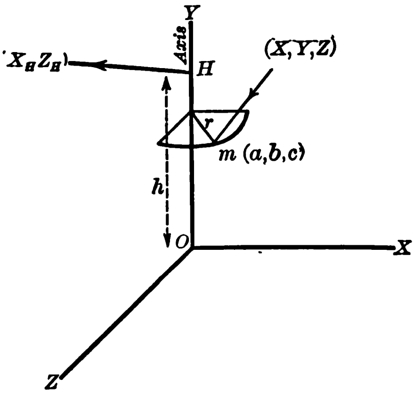


Fig. 56.

Now when an external impulse is applied to a rigid body in this manner, the effect is to give to each small element of mass, m, in the body a certain definite velocity whose components we may indicate by $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$. Accordingly, the entire momentum communicated to the body will have the following components along the axes of X, Y, and Z respectively:

$$\sum\left(m\frac{dx}{dt}\right),\ \sum\left(m\frac{dy}{dt}\right),\ \sum\left(m\frac{dz}{dt}\right).$$

By equating action and reaction, according to Newton's Third Law, we are now able to write the equations of translation,

$$\left.\begin{aligned} X + X_0 + X_h &= \sum\left(m\frac{dx}{dt}\right), \\ Y + Y_0 + Y_h &= \sum\left(m\frac{dy}{dt}\right), \\ Z + Z_0 + Z_h &= \sum\left(m\frac{dz}{dt}\right), \end{aligned}\right\} \text{Eqs. of translation 134.}$$

in which X_0, Y_0, Z_0, X_h, Y_h, and Z_h are six unknown quantities, all the others being given, as data of the problem.

Evidently the solution of the problem is not possible without the aid of more relations connecting the known and unknown quantities. These are obtained by writing the equations of rotation, derived as follows from Newton's Third Law:

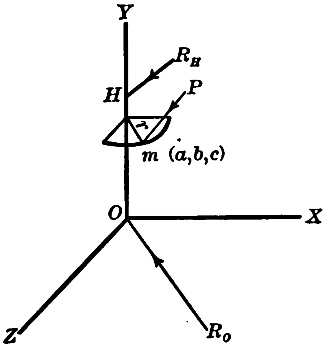


Fig. 57.

Analogue. Equations of Rotation.

Since we have chosen the lower bearing, O, as origin, it is evident that the impulsive reaction (X_0, Y_0, Z_0), applied as it is at O, will produce no angular momentum about *any* axis passing through O; and the same is true of the component Y_h at the upper bearing, for it also acts through the origin, O. The only external forces acting upon the body, then, to produce rotation are the impulses X, Y, Z, X_h, and Z_h, as indicated in Fig. 57. Let us denote the distance between the bearings by h; this will be the arm through which the impulses X_h and Z_h act.

First let us equate the impulsive moment about the axis of X to the angular momentum generated about that axis, remembering that we have agreed (§ 40) to call the right-handed screw relation positive:

$$- Yc + Zb + Z_h h = \Sigma m \left(y \frac{dz}{dt} - z \frac{dy}{dt} \right).$$

Equating momenta about the axes of Y and Z, one obtains in a similar manner

$$\left.\begin{aligned} - Za + Xc \qquad\qquad &= \Sigma m \left(z \frac{dx}{dt} - x \frac{dz}{dt} \right). \\ - Xb + Ya - X_h h &= \Sigma m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right). \end{aligned}\right\} \left.\begin{array}{c} \text{Equations} \\ \text{of rotation} \end{array}\right\} \quad 135.$$

The components of the vector product of linear momentum (mv) and arm (r), i.e. the right-hand member, may be written directly from the determinant by the method of vector analysis as explained in § 37:

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x & y & z \\ \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \end{vmatrix}$$

By remembering that each particle in the body is moving in a circle parallel to the ZX-plane, and that its position coördinate y is therefore a constant, it will be seen that these equations may be greatly simplified.

Let r in Fig. 56 now denote the radius vector in the ZX-plane of *any* particle, m, of the body. Then we have

$$x = r \cos \theta, \quad y = \text{constant}, \quad z = r \sin \theta,$$

where θ is the angle between r and the axis of X measured clockwise from the axis of X.

Differentiating with respect to time, we have, since r is a constant for any one particle, and since θ is a function of the time,

$$\frac{dx}{dt} = -(r \sin \theta)\, \omega = -z\omega,$$

$$\frac{dy}{dt} = 0,$$

$$\frac{dz}{dt} = (r \cos \theta)\, \omega = x\omega,$$

where ω is the angular speed about the Y-axis produced by the impulse P.

Now let $\bar{x}$, $\bar{z}$, denote the horizontal coördinates of the center of mass, and let M indicate the total mass of the body; then since ω is a constant for the entire body, we have

$$\Sigma m \frac{dx}{dt} = -\Sigma mz\omega = -\omega M\bar{z}.$$

$$\Sigma m \frac{dy}{dt} = 0.$$

$$\Sigma m \frac{dz}{dt} = \Sigma mx\omega = \omega M\bar{x}.$$

Accordingly, Eqs. 134 and 135 become

$$\left.\begin{aligned} X + X_0 + X_h + \omega M\bar{z} &= 0, \\ Y + Y_0 + Y_h &= 0, \\ Z + Z_0 + Z_h - \omega M\bar{x} &= 0. \end{aligned}\right\} \text{Translation} \quad \textbf{136.}$$

$$\left.\begin{aligned} Zb - Yc + Z_h h - \omega\Sigma\,(mxy) &= 0, \\ Xc - Za - \omega\Sigma\,(mr^2) &= 0, \\ Ya - Xb - X_h h - \omega\Sigma\,(myz) &= 0. \end{aligned}\right\} \text{Rotation} \quad \textbf{137.}$$

151. The equations of motion are now completely established, and it will be observed that they now contain seven unknown quantities; namely, ω, X_0, X_h, Y_0, Y_h, Z_0, and Z_h. But since Y_0 and Y_h enter symmetrically into a single equation, it will be possible (indeed, it is imperative) for us to treat $Y_0 + Y_h$ as a single quantity. This reduces the number of unknown quantities to six, for the solution of which Eqs. 136 and 137 are entirely competent. For since the fifth equation involves only one unknown quantity, we may solve at once for the angular velocity produced by the blow, namely,

$$\omega = \frac{Xc - Za}{\Sigma\,(mr^2)} = \frac{\text{Moment of Impulse}}{\text{Moment of Inertia}}\,,$$

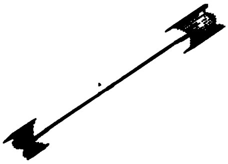
which is now seen to be merely a special case of Eq. 72, § 71.

Introducing this value of ω into the fourth and sixth equations, they will serve to determine the values of Z_h and X_h respectively. Introducing this value of X_h into the first equation, one may solve for X_0; introducing the value of Z_h into the third equation, one obtains Z_0; while the second equation gives the value of $Y_0 + Y_h$, which is the end thrust along the axis. Thus all six of the unknown quantities are definitely determined, and the problem is completely solved.

Problem 7.

152. *To find the center of percussion of a slender rod capable of rotation about a fixed axle.*

The *center of percussion* of a body is that point at which it may be given an impulsive blow, in a direction which is at right angles both to the axis of suspension and to the perpendicular let fall from the given point to the axis, without exerting any impulsive action upon the axle.



the impulse P alone, the point O must move to the left with a velocity v; due to the couple alone, the point O must move to the right with the velocity $\omega \cdot \overline{GO}$.

The sum of these two velocities must be zero. Hence,

$$v - \omega\overline{GO} = 0.$$

What we desire to obtain is the value of $\overline{GA}$, a distance which will locate the point A in the rod. In order to do this, we have merely to substitute, in the last equation, the value of v and ω from Eqs. 138 and 139, thus obtaining

$$\frac{P}{m} - \frac{P \cdot \overline{GA} \cdot \overline{GO}}{I} = 0.$$

Hence, if K be the radius of gyration (§ 102), we have

$$\overline{GA} = \frac{I}{m.\overline{GO}} = \frac{K^2}{\overline{GO}}, \qquad \text{Eq. 140.}$$

an equation which tells us just how far the center of percussion of the rod is from the center of gravity of the rod. The data necessary to compute this distance are evidently the radius of gyration and the distance between the point of suspension and the center of mass.

Every student should roughly verify this result by holding his walking-stick, near the top, between the thumb and forefinger of the left hand, and by then striking the cane with the right hand at a point such that the left hand will feel no shock.

Problem 8.

153. *To find the conditions that the axis of rotation in a rigid body of any shape should experience no impulse when that body receives a blow.*

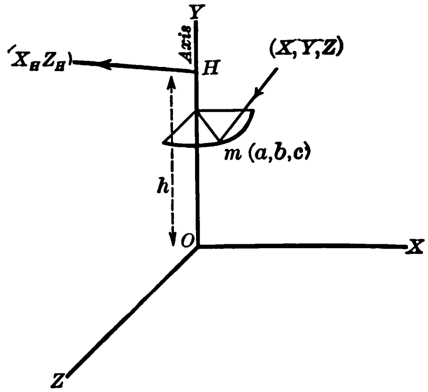


Fig. 56.

To describe this state of affairs, which is merely a special case of Problem 6, we have only to put the two reactions (Fig. 56) of the axle equal to zero; namely,

$$X_0 = Y_0 = Z_0 = X_h = Y_h = Z_h = 0.$$

Our equations of motions (136) then become

$$\left.\begin{aligned} X + \omega M\bar{z} &= 0, \\ Y &= 0, \\ Z - \omega M\bar{x} &= 0. \end{aligned}\right\} \text{Translation Eq. 141.}$$

The condition $Y = 0$ evidently means that the total impulse which can be applied to a body without communication of shock to the axis lies in the XZ-plane. If now we add this condition $Y = 0$ to those just given, we obtain from Eq. 137 the following expressions:

$$\left.\begin{aligned} Zb - \omega\Sigma(mxy) &= 0, \\ Xc - Za - \omega\Sigma(mr^2) &= 0, \\ Xb + \omega\Sigma(myz) &= 0. \end{aligned}\right\} \text{Rotation Eq. 142.}$$

We have now obtained the six conditions which define the position of the center of percussion; but they can be considerably simplified by agreeing that the point at which the blow is struck shall lie somewhere in the XZ-plane; i.e., $b = 0$.

This is quite different from shifting the XZ-plane of reference which has already been fixed as the plane perpendicular to the axle and containing the lower bearing.

Let us also choose the XY-plane such that, at the instant when the blow is struck, it will include both the fixed axis and the center of mass of the body; i.e., $\bar{z} = 0$.

Then the conditions of no shock on the axis, Eqs. 141 and 142, become

$$\left.\begin{aligned} X &= 0, \\ Y &= 0, \\ Z &= \omega M\bar{x}, \end{aligned}\right\} \qquad \text{Translation 143.}$$

and

$$\left.\begin{aligned} \Sigma(mxy) &= 0, \\ Za &= I\omega, \\ \Sigma(mzy) &= 0. \end{aligned}\right\} \qquad \text{Rotation 144.}$$

154. Translating these conditions into words, we have

(1) From the fourth and sixth equations, it follows that the body must be dynamically symmetrical about the XZ-plane, — the plane in which the blow is struck.

(2) From the first and second equations, it follows that the blow must be entirely along the Z-axis; that is, perpendicular to the plane containing the fixed axis and the center of gravity.

(3) The third equation completely determines the value of the angular velocity produced by the blow.

(4) Eliminating $\frac{\omega}{Z}$ between the third and fifth equations we obtain the value of a, the distance from the axis at which the center of percussion lies, namely,

$$a = \frac{I\omega}{Z} = \frac{I}{M\bar{x}}; \qquad \text{Eq. 145.}$$

but this distance a, it will be observed, has exactly the same value as that which we have already found (Eq. 129) for the length of a simple pendulum whose period is the same as that of the rigid body in question when allowed to vibrate, under gravity, about the fixed axis.

The center of percussion is, therefore, identical with the center of oscillation. A ball player, in order not to have the bat "sting" his hands, must see that the ball strikes the bat at the center of oscillation, the point where the hand grips the bat being considered the center of suspension.

Problem 9.

155. *To find the forces exerted by a body spinning about a fixed axis, and acted upon by any forces whatever.*

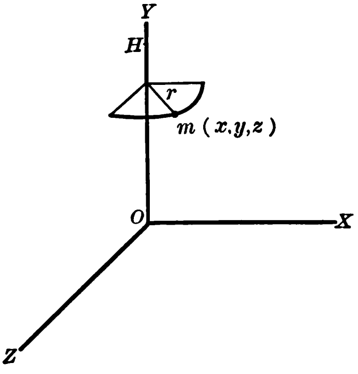


Fig. 59.

As in the two preceding problems, let the axis of rotation be selected as the axis of Y.

Let X, Y, Z be the components of the force applied to any small element of mass at any point (x, y, z) of the body.

Let X_h, Y_h, Z_h, be a force equal and opposite to that exerted by the upper bearing at H; and X_0, Y_0, Z_0, a force equal and opposite to that exerted by the lower bearing at O. Choose the point O as origin, and call the distance $OH = h$; then the equations of motion, according to Newton's Second Law, are Eqs. 107 and 111, § 106, 108.

$$\left.\begin{aligned} X_0 + X_h + \Sigma X &= \Sigma\left(m\,\frac{d^2x}{dt^2}\right), \\ Y_0 + Y_h + \Sigma Y &= \Sigma\left(m\,\frac{d^2y}{dt^2}\right), \\ Z_0 + Z_h + \Sigma Z &= \Sigma\left(m\,\frac{d^2z}{dt^2}\right). \end{aligned}\right\} \text{Translation Eq. 145.}$$

$$\left.\begin{aligned} Z_h h + \Sigma(Zy - Yz) &= \Sigma\left(m\left[\frac{d^2z}{dt^2}\,y - \frac{d^2y}{dt^2}\,z\right]\right), \\ \Sigma(Xz - Zx) &= \Sigma\left(m\left[\frac{d^2x}{dt^2}\,z - \frac{d^2z}{dt^2}\,x\right]\right), \\ -X_h h + \Sigma(Yx - Xy) &= \Sigma\left(m\left[\frac{d^2y}{dt^2}\,x - \frac{d^2x}{dt^2}\,y\right]\right). \end{aligned}\right\} \begin{array}{c}\text{Rotation}\\ \text{Eq. 146.}\end{array}$$

The physical meaning of these equations will be more readily understood when we make the following simplifications:

Since each particle, m, of the body moves in a circle in the XZ-plane, with an angular speed, ω, it is evident that y is a constant. Accordingly,

$$\frac{dy}{dt} = 0, \text{ and } \frac{d^2y}{dt^2} = 0,$$

$$\frac{dx}{dt} = z\omega,$$

$$\frac{dz}{dt} = -x\omega,$$

$$\frac{d^2x}{dt^2} = z\,\frac{d\omega}{dt} + \omega\frac{dz}{dt} = z\,\frac{d\omega}{dt} - x\omega^2,$$

$$\frac{d^2z}{dt^2} = -x\frac{d\omega}{dt} - \omega\frac{dx}{dt} = -x\frac{d\omega}{dt} - z\omega^2.$$

[illegible] substitute these values in the preceding equations, and remember that by definition of the center of mass [illegible])

$$M\bar{x} = \Sigma(mx),$$
$$M\bar{z} = \Sigma(mz),$$

[illegible] following equations, which are more easily interpreted:

$$[illegible] = \frac{d\omega}{dt} M\bar{z} - \omega^2 M\bar{x},$$

$$[illegible] \qquad \text{Eq. 147.}$$

$$[illegible] = \frac{d\omega}{dt} M\bar{x} - \omega^2 M\bar{z};$$

$$\left.\begin{aligned} &[illegible] = \frac{d\omega}{dt}\Sigma(myx) - \omega^2\Sigma(myz),\\ &[illegible] = \frac{d\omega}{dt}\Sigma(m[illegible]) = \frac{d\omega}{dt}\Sigma(mr^2),\\ &[illegible] = \frac{d\omega}{dt}\Sigma(myz) - \omega^2\Sigma(myx). \end{aligned}\right\} \text{Eq. 148.}$$

[illegible] These expressions become very simple if we begin at the fifth equation and interpret them one after another.

Thus the fifth equation tells us that the angular acceleration $\frac{d\omega}{dt}$ of the rotating body at any instant is the ratio of the torque about the Y-axis $\Sigma(Nz - Zx)$, to the moment of inertia with respect to the Y-axis, namely, $\Sigma(mr^2)$.

It will be observed that the forces at the bearings do not enter into this expression. But, of course, if the axis [illegible], as is always the case in practice, and if there is friction at the bearings, this friction will exert a certain torque upon the body. This torque may be taken into account by including friction among the external torques $\Sigma(Xz - Zx)$.

Passing now to the sixth equation, the external forces (X, Y, Z) are known quantities, being a part of the data of the problem. We have, therefore, merely to compute the products of inertia, and integrate the fifth equation for ω, in order to obtain X_h.

In like manner, from the fourth equation we obtain Z_h. It is interesting to observe that X_h and Z_h each vary inversely as h. One can readily see that this ought to be so; for the distance apart of the bearings can make no difference with the motion in case the body is rigid.

$X_h h$ is simply the couple about the Z-axis which we get by introducing two equal and opposite forces $+ X_h$ and $- X_h$ at the origin O, as indicated in Fig. 60. This couple is a definite quantity depending upon the motion of the body, but quite independent of the length of the couple arm, h. This is clear from the fourth and sixth equations.

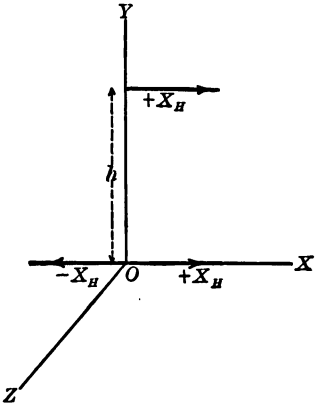


Fig. 60.

Passing now to the second equation, it gives us at once the value of the force along the axis of rotation, namely, $Y_0 + Y_h$, what may be called the "end thrust." By substituting in the first and third equations, the values of X_h and Z_h already found, we obtain X_0 and Z_0, which completes the solution of the problem.

157. *Special Case. A rigid body rotates at a definite angular speed about a fixed axis, under no external forces. Find the forces on the bearings.*

These conditions are the same as the preceding, except that $X = Y = Z = 0$ and ω constant. So that $\frac{d\omega}{dt} = 0$, and the motion of the body is described by Newton's First (instead of his Second) Law. Accordingly, our equations 147 and 148 become.

$$\left.\begin{aligned} X_0 + X_h &= -\omega^2 M\bar{x}, \\ Y_0 + Y_h &= 0, \\ Z_0 + Z_h &= -\omega^2 M\bar{z}. \end{aligned}\right\} \qquad \text{Translation 149.}$$

$$\left.\begin{aligned} Z_h h &= -\omega^2\Sigma(myz), \\ -X_h h &= \omega^2\Sigma(myx). \end{aligned}\right\} \qquad \text{Rotation 150.}$$

The most striking feature of this result is that the resultant of all the forces parallel to the axis of rotation (Y) is zero; in other words, the resultant of the forces on the bearings lies in a plane perpendicular to the axis of rotation. That is, if a body spins about a fixed axis under no external force, the reactions are all at right angles to the axis of spin. It will be observed that the fifth equation completely determines X_h; the fourth equation determines Z_h; substituting these values in the first and third equations respectively, we obtain both X_0 and Z_0.

Again, it is important to observe that the forces on the bearings, in this case, are made up of two components, namely, the centrifugal force and the centrifugal couple, each involving ω^2, and being, therefore, independent of the sense of rotation.

The centrifugal force along the direction of the X-axis is $\omega^2 M\bar{x}$; that along the Z-axis, $\omega^2 M\bar{z}$, corresponding with Eq. 44, § 48. This centrifugal force evidently results from the fact that the center of mass does *not* lie on the axis of rotation; for if it did, we should then have $\bar{x} = \bar{z} = 0$, and each component of the centrifugal force would be zero.

The importance of keeping the center of gravity of a heavy, or rapidly rotating, pulley on the axis will be evident from these equations.

If we simply plot the forces as given in Eqs. 149 and 150 we shall obtain a figure such as Fig. 61. This shows us that even when the centrifugal force is zero, we have left, in general, a centrifugal couple. And it is clear that this centrifugal couple will not tend to translate the axle, but merely to twist or distort it.

Fig. 61, constructed by merely plotting the six forces of Eqs. 149 and 150, shows that there are involved only two forces and two couples, and that these can be combined into a single force and a single couple.

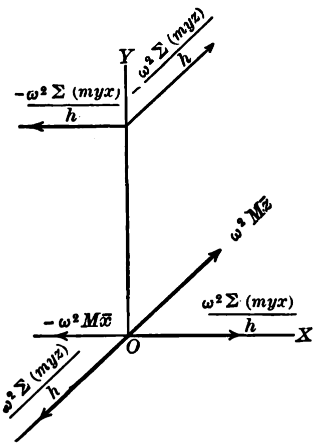


Fig. 61.

It is clear also that in order to eliminate this couple, we have only to select for the axis of rotation a line such that the products of inertia will vanish; that is, such that

$$\Sigma\,(myz) = \Sigma\,(myx) = 0.$$

But these are exactly the conditions for a principal axis of inertia. Hence for a body to be completely "balanced," its axis of rotation must be a principal axis of inertia passing through the center of mass of the body.

When a body is once set into rotation about such an axis as this, the bearings will exert no force on the body, and therefore the motion will remain constant about this one axis; if, however, the body be set spinning about some other axis, even though it pass through the center of mass, there will be a couple acting upon the body, and the motion will *not* remain constant, but will, if the body be free, change to another axis of rotation. These principles will be found of the utmost importance in balancing high-speed machinery.

Problem 10.

158. *Numerical Illustration. The rim of a fly-wheel weighs* 4000 *lbs. and has a diameter of* 6 *ft. The plane of the rim, when keyed to the shaft, is out of true by* 1°; *i.e., the shaft makes an angle of* 1° *with the normal to the plane of the wheel. Find the torque (centrifugal couple) acting upon the bearings when the fly-wheel is rotating at the rate of* 240 *R.P.M. It is assumed that the axis passes through the center of gravity of the wheel.*

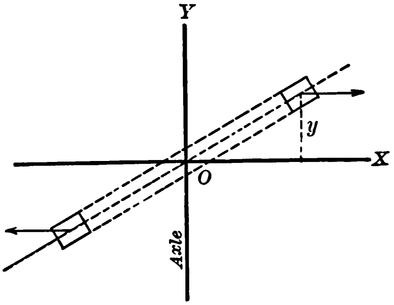


Fig. 62.

The nature of the couple is indicated by the arrows in Fig. 62. Let us choose the axis of rotation for the axis of Y, as in preceding problems. Considering the position of the wheel at any one instant, there will always be one diameter *exactly* perpendicular to the axis of Y. Choose the direction of this diameter for the axis of Z. The axis of X will then, of course, be chosen perpendicular to the YZ-plane. Fig. 62 represents a cross-section through the XY-plane.

Now employ our general equations (150) to compute this couple:

$$\begin{aligned} L &= -Z_h h = \omega^2 \Sigma\,(myz), \\ N &= -X_h h = \omega^2 \Sigma\,(myx). \end{aligned} \qquad \text{Eq. 150.}$$

It is clear that the L component (i.e., the torque about the X-axis) of the couple will be zero for the position we have chosen, since for every value of y on the entire wheel there will be two equal and opposite values of z, so that $\Sigma(myz) = 0$.

As to the component N (i.e., the torque about the Z-axis), it will be observed that for the position which we have selected and for a small angular displacement, θ, such as 1°, we may take $y = x\theta$ instead of $y = x \tan\theta$. And since $x = r\cos\phi$, where ϕ is the azimuth (measured from axis of X) of any particle dM of the rim of the wheel, we have

$$N = \omega^2 \Sigma\,(myx) = \omega^2 \int_0^M yx\,dM = \omega^2 r^2 \theta \int_0^M \cos^2\phi\,dM.$$

And since $dM = \frac{M}{2\pi}\,d\phi$, and since $\int \cos^2\phi\,d\phi = \frac{\phi}{2} + \frac{1}{4}\sin 2\phi$, we have

$$N = \frac{\omega^2 r^2 \theta M}{2\pi} \int_0^{2\pi} \cos^2\phi d\phi = \frac{\omega^2 r^2 \theta M}{2}. \qquad \text{Eq. 151.}$$

Placing $I = Mr^2$ = moment of inertia of rim, we have

$$N = \tfrac{1}{2} I\omega^2\theta = \theta \times \text{kinetic energy of the rim},$$

where θ is measured in radians, and the energy in foot-pounds.

Substituting in Eq. 151 the numerical values of the problem,

M = [illegible] slugs; see § 141.
r = 5 feet.
ω = 8 π radians per second.
θ = [illegible] radians.

the torque about the Z-axis, Fig. 62, becomes

N = 12,000 pound-feet, approximately.

Problem 11.

159. *Another interesting special case of a rigid body rotating about a fixed axis is that in which there are no external forces and no external torques except a couple about the axis of rotation. The problem is to find the condition that there shall be no force exerted on the bearings.*

Let us choose the axis of rotation as the axis of Y, then the data of the problem are expressed algebraically as follows:

$$\Sigma X = \Sigma Y = \Sigma Z = X_h = Y_h = Z_h = 0,$$
$$\Sigma (Zy - Yz) = \Sigma (Yx - Xy) = 0.$$

Among the external forces there remains therefore only the given couple $\Sigma (Xz - Zx)$.

Substituting these values in the general equations of rotation (148), we obtain

$$\left.\begin{array}{r} -\dfrac{d\omega}{dt}\Sigma (myx) - \omega^2\Sigma (myz) = 0, \\ \\ \Sigma (Xz - Zx) - \dfrac{d\omega}{dt}\Sigma (mr^2) = 0, \\ \\ -\dfrac{d\omega}{dt}\Sigma (myz) + \omega^2\Sigma (myx) = 0. \end{array}\right\} \qquad \text{Eq. 152.}$$

Between the first and third of these equations we may eliminate $\frac{d\omega}{dt} \Big/ \omega^2$, and obtain

$$\frac{\Sigma\,(myx)}{\Sigma\,(myz)} = -\,\frac{\Sigma\,(myz)}{\Sigma\,(myx)} \quad \text{or} \quad [\Sigma\,(myx)]^2 + [\Sigma\,(myz)]^2 = 0.$$

Now since the only way in which the *sum* of two squares can be zero is for each of them to be zero, we have

$$\Sigma\,(myx) = \Sigma\,(myz) = 0, \qquad \text{Eq. 153.}$$

as the condition for no pressure on the axes, a condition which holds for all angular speeds and for all angular accelerations. But the condition is precisely the same as that which obtained when the body was entirely free from any torque.

Accordingly, we may describe this result roughly by saying that a couple about the fixed axis does not alter the equilibrium of the body, but merely changes the angular speed at a rate determined by the second of Eqs. 148.

160. NOTE. In a previous discussion (Problem 8, § 153), we have learned how to strike a rigid body mounted on a fixed axis so as to give no impulsive blow to the bearings. Immediately after the rigid body has been struck and has begun to rotate, the forces and couples which we have been studying in Eqs. 149 and 150 are at once called into play. It is important to note, however, that these centrifugal forces and couples are finite, not impulsive. As the simplest possible way of putting these two types of forces into contrast, imagine a baseball suspended by a single thread as shown in Fig. 63, and then given an impulsive blow, *Ft*, in a direction at right angles to the suspending thread, whose length is l. As soon as the blow is struck, a centrifugal force $ml\omega$, at right angles to F, will be called into play; but, like the centrifugal couple, it will not be impulsive.

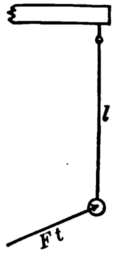


Fig. 63.

R IV.

ION.

brought into contact, a number … are observed, the fundamental … surfaces cannot be moved over … speed, without the exertion of … force is called the *force of* … *friction*. Frictional forces, … vector quantities, and are to be … the other forces. In machinery, … useful feature, at other times a … for instance, it is by friction … pulleys; but, on the other hand, … very large portion of the energy … is wasted.

… generalizations describe the … the behavior of friction:

… slides over any other given … only to a certain limited extent. … the pull (force) on the body is … is called "limiting friction."

… less than this limit, the body … friction will be just equal to … the case may be.

… given surface and material, the … normal force keeping the surfaces together is approximately constant, and is known as the *coefficient of friction.*

This coefficient varies with a large number of circumstances, but, within reasonable limits, it does not vary greatly either with the relative speed of the two surfaces in contact or with the area of these two surfaces.

There is considerable difference between the force required to just set a body in motion and that required to keep it in motion. Oddly enough, the former, which is called statical friction, is considerably larger than the latter, which is called dynamical friction; but Jenkin and Ewing (Phil. Trans. 1877) have shown that there is no discontinuity between these two values, but that if the relative speed be reduced to very minute values, the dynamical friction gradually increases till it equals the statical. In a general way friction between solids is independent of speed, but at high speeds, polishing and abrasion occur so that this rule often ceases to hold. Thus, in the case of a fast railway train, the coefficient of friction between the iron brake-shoe and the tire of the wheel falls to less than half its value for a slowly moving train.

Quantitative and Unambiguous Definition of Friction.

162. When two bodies are in contact throughout any surface, there is a force acting between them, and the direction of this force is, in general, not normal, but oblique, to the surface of contact. If now this force be resolved into two components, one normal to the surface, and the other parallel to the surface, the latter component is known as *the friction.*

When used without qualification, the word "friction" is always understood to mean its maximum value. In the

CHAPTER IV.

FRICTION.

161. When two surfaces are brought into contact, a number of interesting phenomena are observed, the fundamental one being that one of these surfaces cannot be moved over the other, even at uniform speed, without the exertion of force. The resistance to this force is called the *force of friction,* and is said to be due to *friction.* Frictional forces, like all other forces, are vector quantities, and are to be compounded and resolved as are other forces. In machinery, friction is at times a most useful feature, at other times a most wasteful one. Thus, for instance, it is by friction that belts are able to drive pulleys; but, on the other hand, it is through friction that a very large portion of the energy given to an ordinary machine is wasted.

The following experimental generalizations describe the most important features in the behavior of friction:

1. When any given body slides over any other given body, friction exhibits itself only to a certain limited extent. Sliding always begins when the pull (force) on the body is greater than this amount, which is called " limiting friction."

2. If the force on the body is less than this limit, the body will not slide, and frictional reaction will be just equal to the forward pull or push, as the case may be.

3. For any two surfaces of given shape and material, the ratio of this limiting friction to the normal force keeping

the surfaces together is approximately constant, and is known as the *coefficient of friction.*

This coefficient varies with a large number of circumstances, but, within reasonable limits, it does not vary greatly either with the relative speed of the two surfaces in contact, or with the area of these two surfaces.

There is considerable difference between the force required to just set a body in motion and that required to *keep* it in motion. Oddly enough, the former, which is called statical friction, is considerably larger than the latter, which is called dynamical friction; but Jenkin and Ewing (*Phil. Trans.*, 1877) have shown that there is no discontinuity between these two values, but that, if the relative speed be reduced to very minute values, the dynamical friction gradually increases till it equals the statical. In a general way, friction between solids is independent of speed; but, at high speed, polishing and abrasion occur, so that this rule often ceases to hold. Thus, in the case of a fast railway train, the coefficient of friction between the iron brake-shoe and the tire of the wheel falls to less than half its value for a slowly moving train.

Quantitative and Unambiguous Definition of Friction.

162. When two bodies are in contact throughout any surface, there is a force acting between them, and the direction of this force is, in general, not normal, but oblique, to the surface of contact. If now this force be resolved into two components, one normal to the surface, and the other parallel to the surface, the latter component is known as *the friction.*

When used without qualification, the word "friction" is always understood to mean its maximum value. In the third law, above stated, it is, for instance, only this maximum value which is proportional to the normal force.

We may now put this third law into a quantitative and useful form as follows: Let N be the normal force, and F the friction, then

$$F = fN, \qquad \text{Eq. 154.}$$

where f is the proportionality factor, and is generally called the *coefficient of friction.*

But Eq. 154 determines only the *amount* of the force F. In order to obtain the direction, we have also to observe that, *in case the two bodies have any relative motion,* F lies in the same direction as the motion, but has an opposite sense. In case the bodies have no relative motion, we can only say that equilibrium *is* always maintained if it *can* be maintained by the introduction of a frictional force of less than the maximum value.

163. The manner in which the coefficient, f, depends upon lubrication, temperature, pressure, and speed, is described in engineering handbooks.

The following table will at least give an indication of its range of values for various substances:

Material.	f.
Brass on cast iron	0.19
Wrought iron on cast iron	0.20
Wrought iron on wrought iron	0.14
Iron on ice (skates)	0.16 to 0.032
Oak on oak, fibers parallel	0.48
Oak on oak, fibers crossed	0.32
Leather on oak	0.27 to 0.38
Leather on metals, dry	0.56
Leather on metals, wet	0.36
Leather on metals, oily	0.15

ANGLE OF REPOSE. FRICTION CONE.

164. We pass now to two important and helpful conceptions in the actual computation of frictional effects. If a body, *B*, be placed on an inclined plane, as indicated in Fig. 64, its weight may be resolved into two rectangular components, one parallel to the plane, and the other perpendicular to the plane. Let the slope of the plane be variable, and let ϕ be the angle which it makes with the horizontal at any instant; $mg \sin \phi$ will then be the component urging the body down to the plane, while $mg \cos \phi$ will be the normal component. Let us now suppose the angle ϕ to increase until the body, *B*, is just on the verge of slipping, or has just begun to slip, down the plane. At this point, the friction which acts up the plane, reaches its maximum value, and Eq. 154 becomes

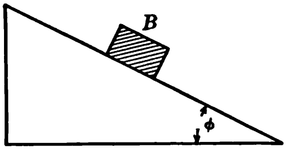


Fig. 64.

$$F = mg \sin \phi = fmg \cos \phi = fN,$$

and hence,

$$f = \tan \phi. \qquad \text{Eq. 155.}$$

The angle, ϕ, thus defined, is known as the *angle of repose.*

If at any point of contact between two surfaces, we erect a normal, the locus of all straight lines passing through this point and making the angle, ϕ, with this normal will be a circular cone, having its apex at the point of contact. This is called the *cone of friction.* Its usefulness will be evident from the following examples:

Then if the box is to slide, and not rotate,

$$Wx > Fy,$$

or

$$Wx > fWy.$$

Hence $\frac{x}{y} > f$, and hence also $> \tan\ \phi$, a result which is identical with graphical solution obtained above.

Friction at Bearings. Friction Circle.

167. When an axle runs in a bearing, the surface of the shaft slides over the surface of the bearing, and the axle generally carries, in addition to its own weight, a certain load. The total force which the axle thus exerts upon the bearing we shall denote by Q. This force, Q, does not denote merely the weight of the shaft, but the resultant of the external forces acting upon the shaft, such, for instance, as the pull of a belt, or the weight of a fly-wheel. When the shaft is at rest, this force is balanced by the normal force which the elements of surface of the bearing exert upon the axle, as indicated in Fig. 67. Let us denote these various reactions by $\overline{N}$; then by Newton's Third Law, we have as the equation of equilibrium,

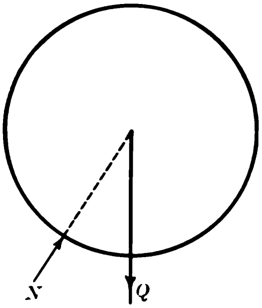


Fig. 67.

$$\Sigma \overline{N} + \overline{Q} = 0.$$

This, it will be observed, is a vector equation, the vector sum of the $\overline{N}$'s being equal and opposite to the single vector,

$\overline{Q}$; and the resultant of these two forces is zero, because the center of gravity of the shaft does not move.

But now let us denote by N, without a vinculum, the merely scalar part of the normal force. Do the same for Q. Then

$$\Sigma N \underset{>}{=} Q.$$

For instance, the bearing may be clamped upon the axle, by means of set screws, in such a way that the total normal pressure, ΣN, is many times greater than the external force, Q. Accordingly, in general we have

$$\Sigma N = \alpha Q, \qquad \text{Eq. 156.}$$

a scalar equation which defines α, a coefficient which in practice is always greater than unity.

168. In what precedes, we have assumed that the force on the bearing passes through the center of the bearing, and hence exerts no torque upon the shaft. Let us now assume that the external force, Q, is somewhat eccentric, as indicated in Fig. 68. Let q be the distance from the center at which the line of action of Q passes through the shaft.

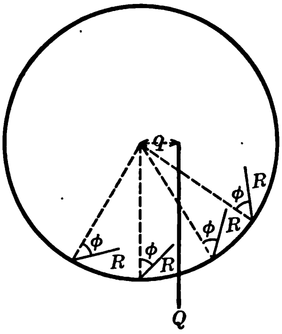


Fig. 68.

Such an external force as this may be brought into play by means of a belt acting on a pulley, the tension on the two sides of the belt being unequal.

It is evident that under these circumstances, *if there be no friction*, the axle will rotate in response to the torque, $\overline{Q}\,\overline{q}$, because the resultant of any number of purely normal forces must pass through the center of the shaft, and this resultant will therefore have no torque about the center of the shaft. Under the assumption of no friction, the rotation will, of course, be an accelerated one.

Problem 14.

169. *We now pass to the case met with in actual practice, namely, (i) where friction is present and (ii) the external force is eccentric. The problem is to find the torque required to just start such a shaft into rotation.*

In Fig. 68 let $\overline{Q}$ represent the external force, and $\overline{R}$ the force on each element of surface of the axle. Evidently this force, $\overline{R}$, is the resultant of the normal force, N, and the friction, F. If the axle is just on the point of turning, the angle between N and R, which we have indicated by ϕ, will be the semi-angle of the friction cone for that pair of surfaces.

Since the center of gravity of the shaft does not move, it is clear that $\Sigma\overline{R} + \overline{Q} = 0$, which is the condition for no linear acceleration — in this case no translation.

The condition that the shaft shall be just on the verge of rotation is that the moment of the external force, Q, about the center shall be just equal to the maximum value of the moment which the frictional components, F, exert about the center. It is clear that the normal components, N, exert no moment about the center of the shaft, since they all pass through this point.

Let r denote the radius of the shaft, then the equation of rotational equilibrium,

$$r\Sigma F - Qq = 0, \qquad \text{Eq. 157.}$$

becomes the defining equation for q.

Using the coefficient of friction f and Eq. 156, we may write

$$r\Sigma F = fr\Sigma N = fraQ = \rho Q$$

or

$$\rho = fra = [illegible]. \qquad \text{Eq. 158}$$

a being the quantity defined above and always greater than unity.

The quantity, ρ, thus determined is the radius of a circle which is concentric with the shaft and which is known as the *friction circle.*

The solution of the problem is therefore contained in Eqs. 157 and 158, between which ρ may be eliminated and Q evaluated.

The friction circle has the following important property, namely, so long as the shaft is at rest, the direction of the reaction R (see Fig. 69), falls inside of the circle. But so long as the shaft rotates uniformly, the direction of this reaction is just tangent to the circle, since the friction here assumes its maximum value.

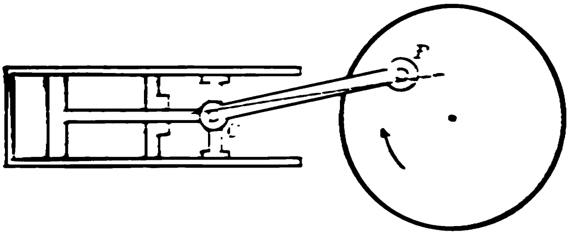


Fig. 69.

170. As an extension of this problem, and as an illustration of the usefulness of the friction circle, consider the connecting rod of a locomotive. If there were no friction at either end, the direction of the thrust would be alc joining the centers of the cross-head, C, and the crank pin, P. But in consequence of friction this direction of thrust is shifted to coincide with a line which is tangent to the *lower* side of the friction circle at the crank pin, and the *upper* side of the friction circle at the cross-head. Knowing the radii of these friction circles, one can easily compute the amount by which friction diminishes the moment of the thrust of the piston rod.

Diminution of Friction at Bearings. Rolling Friction.

171. *Ball Bearings.* The friction of an ordinary shaft in an ordinary bearing is, of course, sliding friction. One of the most familiar and effective methods for reducing the energy-waste in such a bearing is to substitute, for sliding friction, the *rolling friction* which the shaft encounters when made to travel upon a row of steel balls as indicated in Fig. 70.

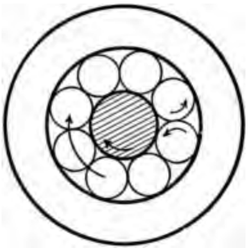


Fig. 70.

It is well known that the resistance which a wheel encounters in rolling over an ordinary road is much less than that of the same wheel sliding over the same road. The diminution of friction in the case of ball bearings is of the same kind. The rotation and the travel of the balls are indicated in Fig. 70. It will be observed, however, that even here the sliding friction is not entirely eliminated, since each two adjacent balls are continually sliding one past the other. The balls, however, fit very loosely into their track; accordingly, the lateral pressure (i.e., the pressure at right angles to the radius of the shaft) is very slight, so that this sliding friction absorbs very little energy.

172. *Friction Wheels.* Another method of avoiding sliding friction is to carry the shaft on a pair of wheels such as those indicated by A and B in Fig. 71. This has the double advantage of reducing all the friction of the wheel, C, to rolling resistance, and of reducing the power wasted to that absorbed by two slowly moving bearings.

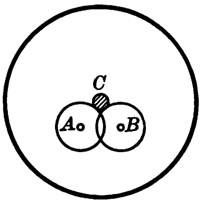


Fig. 71.

173. *Step Bearing.* Another effective device for diminishing friction is that in which the shaft and its load are supported on the end of the shaft, S, the shaft standing in a vertical position, and resting upon a thrust block, B, as indicated in Fig. 72. This form of bearing is known as the *step bearing,* and is widely employed in modern turbine engines where the speed is so high that the work of friction would be enormous if the coefficient, f, were not extremely small. Here the resistance is still that of sliding friction; but it is immensely reduced in the case of the steam turbine by pumping oil in at A under such pressure (generally over 1000 lbs. to the square inch) that the shaft is lifted off the thrust block and rotates upon a film of oil. For the more complete theory of this device (in which the relative speed of the sliding surfaces is not constant, but increases from the center of the shaft out), the student is referred to the engineering handbooks, such as Stodola's *Steam Turbine,* or Föppl's *Technische Mechanik.* The same reference must also be made for a discussion of rolling friction, and the friction of belts, ropes and screws.

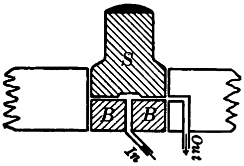


Fig. 72.

Work of Friction.

174. It is well known that all the work employed in overcoming friction is transformed into heat.

It was shown by Joule, the English physicist, more than half a century ago, that the ratio of the work thus lost to the heat produced is a constant. In Engineer's units this constant (which is known as "the mechanical equivalent of heat," and is one of the most important constants of nature) has the numerical value of 772 approximately; i.e., 772 foot-pounds of work are required to raise the temperature of one pound of water one degree Fahrenheit. In the C. G. S. system, this constant is 41.87×10^6; that is, approximately 42 million ergs of work are required to raise the temperature of one gram of water one degree Centigrade. The specific heat of water varies slightly with the temperature, and the more exact value given above 41.87×10^6 applies to water between 15° and 16° C.

To compute the work of friction between any pair of rubbing surfaces, we have merely to multiply the force of friction by the distance through which it is exerted. Thus, in the case of translation where N is the load carried, f the coefficient of friction, and S the distance,

$$\text{Work} = W = f\int_0^S N\,ds,$$

which is merely a special case of the defining equation (112) for work.

If we denote the relative speed of the sliding surfaces by v, the power, P, absorbed by friction will be, Eq. 123,

$$P = \frac{dW}{dt} = fN\frac{ds}{dt} = fNv = \text{Force of friction} \times \text{speed}. \qquad \text{Eq. 159.}$$

Analogue.

In the case of rotation, we obtain the loss of power in a strictly analogous way: by multiplying the *moment of friction* by the relative angular speed of the two surfaces. To obtain the moment of friction, say on the thrust block (or step) of a step bearing, let us denote by da an element of area of the surfaces in contact. Let p denote the normal pressure at this point. The normal force will then be $p\,da$; the force of friction over this area will be $fp\,da$. If r be the distance of this element of area from the center, the moment of friction, for this element, will be $frp\,da$. Hence,

$$\text{Total moment of friction} = f\int_0^A rp\,da = L, \text{say.}$$

Let ω denote the angular speed of one surface over the other, then the *power* absorbed by friction is

$$P = L\omega = f\omega\int_0^A rp\,da. \quad \text{Eq. 160.}$$

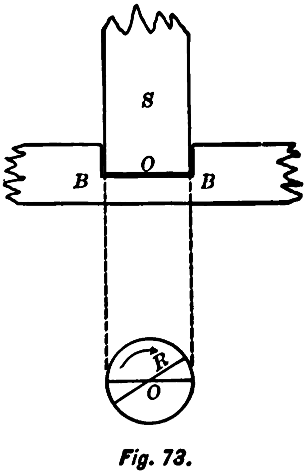


Fig. 73.

There are two interesting special cases of this general expression for the power absorbed at one surface rotating over another.

175. *Case I. The Step Bearing.* When one has to deal with a solid flat pivot bearing such as that represented in Fig. 73, he may consider the pressure, p, as given by the following equation where W

denotes the weight of the shaft and its load, and R is the radius of the bearing surface:

$$p = \frac{W}{\pi R^2} = \text{constant over entire bearing surface.}$$

Introducing this value of p into Eq. 160, and placing

$$da = 2\,\pi r\,dr,$$

where r is the distance from the axis to any element of area on the bearing surface, one has for the power absorbed,

$$P = 2\,\pi f\omega p \int_0^R r^2\,dr = \frac{2\,Wf\omega}{R^2}\int_0^R r^2\,dr = \frac{2}{3}\,Wf\omega R. \qquad \text{Eq. 161.}$$

This result shows how the power wasted in heat varies with the radius of the pivot.

The student will find it an interesting exercise to compute a corresponding expression for the case of a hollow flat pivot, such as that represented in Fig. 72.

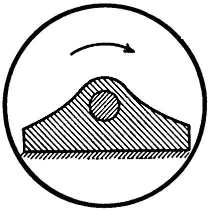


Fig. 74.

176. *Case II.* When one has to deal with a horizontal cylindrical shaft rotating in a cylindrical bearing such as that shown in Fig. 74, the conditions are somewhat changed; for here r is a constant, and may, therefore, be placed outside the integral sign; while the load and pressure are related as follows:

$$\int_0^A p\,da = \text{load} = B,$$

where A is the total area of the bearing surface.

Introducing this value into the general equation (160), we have

$$P = f\omega r\int_0^A p\,da = f\omega rB, \qquad \text{Eq. 162.}$$

which is the working equation for the friction in a bearing such as that of a dynamo armature. It should be carefully noted that the integral of $p\,da$ may be very much greater than the mere weight, say, of the shaft and armature, or even of the weight of the armature plus the pull of the driving belt; for, if the journal be too small or too tightly clamped, the value of p may be enormously increased.

THE PRONY DYNAMOMETER.

177. One of the most interesting instances of the work done by frictional forces is that employed in the instrument devised by Prony for measuring the rate at which a machine is doing work, and known as the Prony Dynamometer, sometimes called the Prony Brake.

In Fig. 75 let A represent the fly-wheel of an engine over which is thrown a strap fastened at one end to the bed of the engine, and carrying at the other end a weight W. The spring balance S is inserted near the fixed end of the belt. When the wheel is at rest, the tension T_1 may be equal to tension T_2; but when the wheel rotates inside this belt at a uniform rate, the moment of force which the engine exerts upon the wheel is exactly equal and opposite to the moment of force exerted by the frictional resistance F; but

$$F = T_2 - T_1 = W - T_1,$$

where T_1 is the reading of the spring S, in pounds. The tension, it will be observed, is always greater on the side of the wheel which is moving in the direction which is opposite to that of the tension. The torque exerted by the frictional

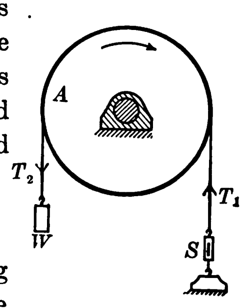


Fig. 75.

force F is Fr, where r is the radius of the wheel; and since there is equilibrium, it must follow that

$$Fr = (W - T_1)r.$$

If the wheel makes p revolutions a minute, then the work done by the engine during each minute will be $2\pi pFr$.

If F and r be measured in pounds and feet respectively, this result will, of course, be, in foot-pounds per minute,

$$\text{H.P. of engine} = \frac{2\pi pFr}{60 \times 550} = \frac{2\pi pFr}{33{,}000}. \qquad \text{Eq. 163.}$$

FLUID FRICTION.

178. The resistance which a solid meets in moving through a fluid is quite different from that encountered in moving over another solid. The reason for this will be clear when it is considered that a solid moving through a fluid sets the parts of fluid into relative motion, and thus confers upon them kinetic energy, which, owing to the viscosity of the fluid, is frittered away in heat. This is true both of liquids and gases. For this reason it is found, experimentally, that fluid friction is no longer independent of speed, as is practically the case when one solid body moves over another. If the speed of the solid, with reference to the fluid, is small, experiment shows that the resistance varies nearly as the first power of the velocity. If, however, the velocity be great, such as that of an ocean liner, the resistance increases as the second and higher powers of the velocity. A fuller discussion of this subject must be sought in engineering treatises on Hydraulics.

A brief but excellent summary of the facts of the case is contained in the following table from Perry's *Applied Mechanics*, page 80.

COMPARISON OF THE LAWS OF FLUID AND SOLID FRICTION.

Friction between Solids.	Fluid Friction.
1. The force of friction does not much depend on the velocity, but is certainly greatest at slow speeds.	1. The force of friction very much depends on the velocity, and is indefinitely small when the speed is very slow.
2. The force of friction is proportional to the total pressure between two surfaces.	2. The force of friction does not depend on the pressure.
3. The force of friction is independent of the areas of the rubbing surfaces.	3. The force of friction is proportional to the area of the wetted surface.
4. The force of friction depends very much on the nature of the rubbing surfaces, their roughness, etc.	4. The force of friction at moderate speeds does not much depend on the nature of the wetted surfaces.

Friction in Springs. Damped Oscillations. Simple Harmonic Motion Modified by Friction.

179. When an oscillating system is left to itself, and is not driven by any external source of energy, its vibrations are said to be *free;* but, when the oscillations of the system are maintained by an external periodic force, its vibrations are said to be *forced.*

In actual practice, both forced and free vibrations are, of course, subject to damping by friction, the difference being that the amplitude of the free vibration is sooner or later reduced to zero, while that of the forced vibration is merely diminished, not extinguished, by damping. That important type of motion, called "Simple Harmonic," which we have already studied to some extent, is one which never occurs in nature, for the reason that friction of some kind *always* interferes so as to continually reduce the amplitude of the swing. Motions which are close approximations to Simple Harmonic are, however, encountered daily. Such are the motions of a plucked guitar string, a bridge girder from which a load has been suddenly removed, a liquid surface traversed by waves, etc. We shall consider first free and afterwards forced oscillations.

Free Oscillations of a Loaded Spiral Spring.

180. As a type of one of the most frequent and important of free vibrations, consider a vertical spiral spring, S, loaded with a mass, m. At the lower end of the mass imagine a small piece of metal sheet attached and dipping into a vessel filled with oil. If now this mass, m, be displaced in a vertical direction from its position of equilibrium, there will be called into play two, and only two, external forces. Let us suppose the mass, m, is displaced downwards and then released. Let the elongation at any instant, t, be denoted by y. When y is negative (i.e., when the displacement is downwards), there will, according to Hooke's Law, be called into play a force of $+a^2y$ dynes acting upward where a^2 is the constant of the spring, i.e., the force required to produce unit displacement of the spring. When the mass, m, reaches its position of static equilibrium, it will not stop, but will move on upward until its motion is gradually destroyed by the excess of its weight over the pull of the spring, and will then proceed to repeat the motion just described, only in the reverse direction.

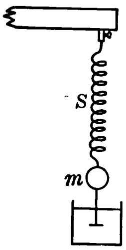


Fig. 76.

Since Hooke's Law applies to the shortened as well as the elongated spring, it is clear that when the mass, m, is at a distance $+ y$ above its position of equilibrium, the excess of the weight over the pull of the spring will be $- a^2y$ dynes acting downward. And so, in general, we may say that whenever the mass is displaced by an amount y, there is a restoring force $- a^2y$, where a^2 is a constant which is always positive.

In addition to this, there is the frictional force exerted by the oil upon the lower end of the spring and proportional to the velocity. This force is, like other frictional forces, always in the same direction as the velocity of the spring, but opposite in sense; so that we may place it equal to $- r \frac{dy}{dt}$, where r is a positive constant equal to the frictional force brought into play when the wire moves through the oil with unit speed.

As we have already learned (Eq. 108), the equation which describes the translational motion of any rigid body is obtained by equating the sum of the external forces to the mass-acceleration of the body. Accordingly we have, if we denote each of the external forces by F,

$$\Sigma F \equiv - a^2y - r \frac{dy}{dt} = m \frac{d^2y}{dt^2},$$

an equation of motion which would accurately describe the motion of the mass, m, if the spring itself had no inertia. It has been shown, Problem 5, § 149, that in order to take account of the mass of the spring, which we may call S, we have only to add to m one third of the mass of the spring. Thus, putting $m + \frac{S}{3} = M$, we have for our working differential equation,

$$M \frac{d^2y}{dt^2} + r \frac{dy}{dt} + a^2y = 0. \qquad \text{Eq. 164.}$$

But before we can use this equation in the laboratory for purposes of computation, we must first integrate it; i.e., obtain from it an integral relation between y and t.

We shall first prove that the following integral relation is a particular solution of the differential equation; namely,

$$y = \varepsilon^{\lambda t}, \qquad \text{Eq. 165.}$$

where ε = logarithmic base 2.71828 . . ., and λ is a quantity to be determined by substituting this value of y in the given differential equation. Differentiating, we find,

$$\frac{dy}{dt} = \lambda \varepsilon^{\lambda t} = \lambda y, \text{ and } \frac{d^2y}{dt^2} = \lambda^2 \varepsilon^{\lambda t} = \lambda^2 y.$$

Substituting these values for the differential coefficients in Eq. 164, we see that Eq. 165 is a solution provided

$$\lambda^2 + \frac{r}{M}\lambda + \frac{a^2}{M} = 0. \qquad \text{Eq. 166.}$$

Any value of λ which satisfies 166 will, when substituted in 165, give us a solution of 164.

The possible values of λ are the roots of the quadratic (166), and these are,

$$\lambda = \frac{-r}{2M} \pm \sqrt{\frac{r^2}{4M^2} - \frac{a^2}{M}}. \qquad \text{Eq. 167.}$$

We thus have two values of λ at our disposal, namely, λ_1 and λ_2; so that by using them each as particular solutions, and introducing two constants of integration, A and B, we have (since the differential equation is linear and of the second order), for a complete solution,

$$y = A\varepsilon^{+\lambda_1 t} + B\varepsilon^{+\lambda_2 t}. \qquad \text{Eq. 168.}$$

But this pair of values for λ may be either real or complex according as the radical in 167 is real or imaginary; and, since the physical meaning of the solution is very different in these two cases, we shall consider them separately.

Aperiodic, or "Deadbeat," Motion.

181. If the damping resistance, r, of the oil upon the metal sheet, which dips into it (Fig. 76), is sufficiently large to make

$$\frac{r}{2M} > \frac{a}{\sqrt{M}}, \qquad \text{Eq. 169.}$$

we shall have the folowing real values for λ:

$$\lambda_1 = -\frac{r}{2M} + \sqrt{\frac{r^2}{4M^2} - \frac{a^2}{M}}, \qquad \text{Eq. 170.}$$

$$\lambda_2 = -\frac{r}{2M} - \sqrt{\frac{r^2}{4M^2} - \frac{a^2}{M}};$$

and since the radical is always less than the first term of the right-hand member, it is clear that *λ_1 and λ_2 are each essentially negative,* and that *the numerical value of λ_2 is greater than that of λ_1,* a fact which may be indicated by the following inequality:

$$|\lambda_2| > |\lambda_1|.$$

To complete the solution, it remains to determine the constants of integration, A and B, in Eq. 168.

To do this, we have only to remember that at the instant $t = 0$ the spring m must have been in some definite position, say y_0, and must have been moving with some definite velocity, say v_0. Putting $t = 0$, in Eq. 168, we shall then have,

$$y_0 = A + B.$$

In like manner, $v_0 = \left(\frac{dy}{dt}\right)_{t=0} = \lambda_1 A + \lambda_2 B.$

Solving these equations for A and B, we have

$$A = \frac{\lambda_2 y_0 - V_0}{\lambda_2 - \lambda_1},$$

$$B = -\frac{\lambda_1 y_0 - V_0}{\lambda_2 - \lambda_1}.$$

The solution, $$y = A\varepsilon^{\lambda_1 t} + B\varepsilon^{\lambda_2 t}, \qquad \text{Eq. 168.}$$

is now complete; for all the four constants have been determined in terms of the data of the spring. There remains only the interpretation.

Each term in the right-hand member of 168 represents a displacement of the spring which is continually decreasing, for λ_1 and λ_2, it will be remembered, are each negative constants. Accordingly if A is positive and B negative, and if $A > B$, we may represent these two terms by the upper and lower dotted curves respectively, of Fig. 77, where the abscissa denotes the time and the ordinate the displacement of the spring from its position of equilibrium.

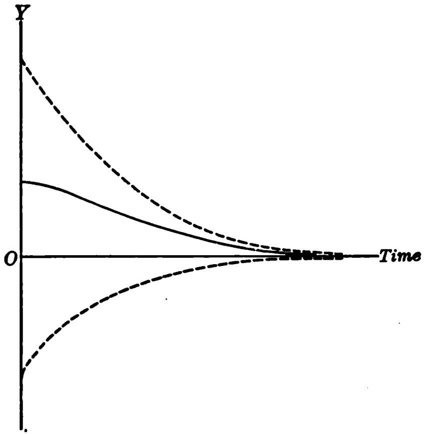


Fig. 77.

The full curve is the resultant of the two dotted curves, obtained by adding their ordinates, and therefore represents the complete solution, and shows us how a spring will recover its position of equilibrium when the damping is considerable.

If $B > A$, then the two curves would be represented by the dotted lines in Fig. 78. Here it will be seen that the spring when released passes once through its position of equilibrium, reaches a maximum, and approaches its place of rest in an asymptotic manner.

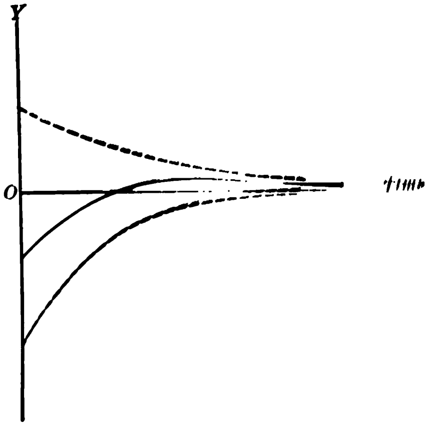


Fig. 78.

Plot the resultant curve when A and B have the same sign. The student will find it an interesting exercise to prove that the spring will recover equilibrium in the least possible time when the radical in Eq. 170 disappears; i.e., when

$$\frac{r}{2M} = \frac{a}{\sqrt{M}}$$

We now proceed to consider the second case, namely, when the damping is smaller, so that the radical in Eq. 170 is imaginary; i.e., when

$$\frac{r}{2M} < \frac{a}{\sqrt{M}};$$

This leads us to the case of

Damped Periodical Vibrations.

182. The roots of Eq. 166 are now complex, so that we may write

$$\lambda = \frac{r}{-2M} \pm i\sqrt{\frac{a^2}{M} - \frac{r^2}{4M^2}}, \qquad \text{Eq. 171.}$$

where i denotes, as usual, $\sqrt{-1}$.

For the sake of brevity, let us now define b and n as follows:

$$\left.\begin{aligned}\frac{r}{2\,M} &= b,\\ \sqrt{\frac{a^2}{M} - \frac{r^2}{4\,M^2}} &= n.\end{aligned}\right\}$$

Then we have

$$\lambda = -\,b \pm in,$$

and hence Eq. 168 becomes

$$\begin{aligned} y &= A\,\varepsilon^{(-b+in)t} + B\,\varepsilon^{(-b-in)t}\\ &= \varepsilon^{-bt}[A\,\varepsilon^{+int} + B\,\varepsilon^{-int}.] \end{aligned} \qquad \text{Eq. 172.}$$

It remains now to determine the constants A and B in terms of the initial conditions.

Let the initial velocity of the spring be zero. Then

$$\frac{dy}{dt} \equiv 0 = A\,(-\,b + in) + B\,(-\,b - in). \qquad \text{Eq. 173.}$$

Also, if y_0 be the initial position of the spring, we have

$$y_0 = A + B. \qquad \text{Eq. 174.}$$

Solving 173 and 174 for A and B, we obtain

$$A = -\left[\frac{(-\,b - in)}{2\,in}\right]y_0,$$

$$B = +\left[\frac{(-\,b + in)}{2\,in}\right]y_0.$$

Substituting these values in the second equation (172), and remembering that

$$\varepsilon^{+int} = \cos nt + i\sin nt,$$

$$\varepsilon^{-int} = \cos nt - i\sin nt,$$

we have

$$y = -\,\varepsilon^{-bt}\left\{+\left[\frac{-\,b - in}{2\,in}\right]y_0\,(\cos nt + i\sin nt)\right.$$

$$\left. -\left[\frac{-\,b + in}{2\,in}\right]y_0\,(\cos nt - i\sin nt)\right\}.$$

Up to this point y has been a complex quantity, made up of a real and an imaginary part.

Let us now equate the *real* parts of the two members of the equation; we shall then have

$$y = + y_0 \varepsilon^{-bt}(\cos nt - \frac{b}{n} \sin nt).$$

Our solution is now complete: we have an integral relation giving y in terms of t, with the constants of integration each determined. This expression may, however, be transformed into another which makes the physical meaning much clearer. Let us define α by the following equation:

$$\alpha = \tan^{-1}\frac{b}{n}.$$

Then,

$$\frac{b}{\sqrt{b^2 + n^2}} = \sin \alpha,$$

$$\frac{n}{\sqrt{b^2 + n^2}} = \cos \alpha,$$

and

$$\begin{aligned} y &= y_0 \varepsilon^{-bt} \frac{\sqrt{b^2 + n^2}}{n} [\cos \alpha \cos nt - \sin \alpha \sin nt] \\ &= A \varepsilon^{-bt} \cos (nt + \alpha) \qquad \text{Eq. 175.} \end{aligned}$$

$$\text{where} \begin{cases} A = y_0 \dfrac{\sqrt{b^2 + n^2}}{n}, \\ \alpha = \tan^{-1}\dfrac{b}{n}, \\ b = \dfrac{r}{2M}, \\ n = \sqrt{\dfrac{a^2}{M} - \dfrac{r^2}{4M^2}}, \\ \text{Period} = T = \dfrac{2\pi}{n} = \dfrac{2\pi}{\sqrt{\dfrac{a^2}{M} - \dfrac{r^2}{4M^2}}}. \end{cases}$$

Eq. 175 will be recognized as a motion which is Simple Harmonic, except that the amplitude is continually diminishing with the time.

From the value of T it is evident that the period of the spring is constant, but somewhat greater than it would be if there were no damping; i.e., if r were zero. For the period of an undamped spring is $2\,\pi\sqrt{\frac{M}{a^2}}$, where a^2 represents the constant of the spring. See Eq. 48, § 49.

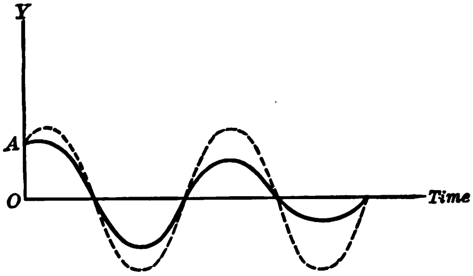


Fig. 79.

Graph of Eq. 175 with and without Damping Factor.

And since $\frac{r^2}{4\,M^2}$ is always positive, the effect of the damping, measured by r, must always be to increase the period. It should be observed also that the rapidity of damping, as measured by the factor ε^{-bt}, is a function of r, since $b = \frac{r}{2\,M}$. The manner in which this factor, and hence the amplitude, varies with time, is shown in Figs. 77 and 78.

Fig. 80 will make clear the relation of the damped harmonic curve and the exponential curve discussed in the last section.

It is important to observe that, while the period of the damped oscillation is slightly different from that of the undamped, there is no difference between the period of the cosine in Eq. 175 and the period of the damped vibration.

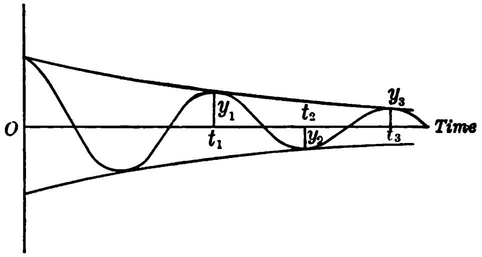


Fig. 80.

In other words, the frequencies of the two curves in Fig. 79 are exactly equal; for the dotted curve of Fig. 79 does *not* represent the free vibration of this spring which we have been discussing, but the free vibration of a spring having an undamped period equal to the damped period of our spring.

Experimental Determination of Damping. Logarithmic Decrement.

183. Referring to Fig. 80, and remembering that the period of the trigonometric factor is the same as the period of the damped oscillation, we see that the cosine factor will have the same value at each maximum displacement on the same side of the time axis. And at any two successive maximum displacements the cosine will also have the same value; but from one side to the other it will differ in sign. Hence for the a^{th} and the $(a + 1)^{\text{th}}$ vibration, we shall have

$$\frac{y_{a+1}}{y_a} = \frac{\varepsilon^{-(a+1)b\frac{T}{2}}}{\varepsilon^{-ab\frac{T}{2}}} = \varepsilon^{-b\frac{T}{2}} = \textit{constant.}$$

If, then, our original supposition, that the force of damping is proportioned to the speed, is correct, we should find that the ratio between any two successive elongations of the spring is a constant. This result has been verified in very many cases by experiments with damped vibrations of various kinds.

Since the successive maximum ordinates form a geometric series, it is clear that the logarithms of these maxima will form an arithmetical series.

If we employ natural logarithms, we may therefore write

$$\log y_{(a+1)} - \log y_a = -b\frac{T}{2} = \textit{constant}. \qquad \text{Eq. 176.}$$

This is essentially the laboratory equation for determining b; for since y_{a+1}, y_a and T are each measurable, we may use the equation to find b.

The constant $b\frac{T}{2}$ or $\log\frac{y_{a+1}}{y_a}$ is known as the logarithmic decrement, and may be defined as *the natural logarithm of the ratio of two successive elongations of the spring.*

Going back to our definition of b, the connection between this logarithmic decrement and the frictional resistance r will be clear; namely,

$$\log\frac{y_{a+1}}{y_a} = \frac{bT}{2} = \frac{rT}{4M}. \qquad \text{Eq. 177.}$$

The rate at which a spring " dies down " depends, therefore, in this manner, upon the *friction*, the *period*, and the *load*. See excellent and simple treatment of " Forced Vibrations " in Webster's *Dynamics*, page 152. See also Perry's *Applied Mechanics*, pages 613–618 (ed. 1899).

Problem 15.

184. *A rigid body is rotating about a given horizontal axis (say that of Z) with a constant angular speed Ω. Its moment of inertia about this same axis is I. Find what torque will be required to rotate the shaft of the body about a vertical axis (say that of Y) with a uniform angular speed ω. Neglect friction and determine the torque both in direction and amount.*

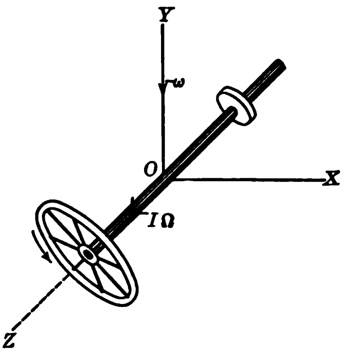


Fig. 81.

Before we consider this problem in detail, it will be well to recall the general principle that forces and torques respectively are obtained by multiplying linear and angular inertias by their corresponding accelerations. Of course this general principle really amounts to nothing but the definition of force.

SHORT DIGRESSION ON CENTRIFUGAL FORCE AND PRECESSIONAL COUPLE.

185. In Eq. 44, § 48, we have seen that when a particle moves in a circle with uniform linear speed, and when the radius, joining the particle with the center of the circle, rotates with an angular speed ω, there must be a linear acceleration of ωv along the radius and towards the center; and since the particle moves with uniform speed in its path, there is no tangential acceleration; this normal acceleration ωv is therefore the total acceleration.

If the inertia of the particle is m, the force required to hold the particle in its circular path is $m\omega v$, a quantity which has received the name of " centrifugal force."

It will be well, however, to look at this matter also from the vector viewpoint, which is as follows:

The essential features of this motion are important. They are,

(i) We do not now inquire as to how the motion originated. We merely assume a particle in uniform circular motion, such practically as that which is met with in the moon revolving about the earth.

(ii) We have a particle whose linear momentum (a vector quantity) is being *changed in direction* at the uniform rate of ω radians per second; this angular velocity ω is a vector, and is always at right angles to the vector mv.

(iii) We have (as a necessary consequence) a force whose direction is at right angles to each of the preceding vectors, and whose amount is equal to their product; namely, $(mv)\omega$. See Eq. 44.

Let the dotted circle in Fig. 82 represent the path of the particle m. Choose the plane of the circle as the XZ-plane, and the center of the circle as origin.

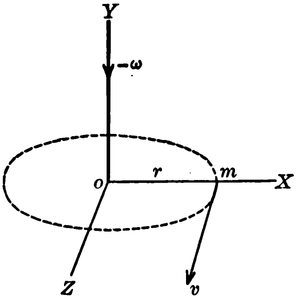


Fig. 82.

Take the axis of X in the direction in which $\overline{Om}$ lies at the instant under consideration. Then the momentum of the particle $\overline{mv}$ will lie along the direction of the Z-axis, and angular velocity ω will lie along the Y-axis. And if r and v are each taken positive, as indicated in Fig. 82, ω will be in the left-handed screw relation; i.e., ω will be negative.

Since now the centrifugal force, F, is the vector product of these two quantities, we have

$$F = -[mv.\omega] \text{ acting along the axis of } X. \quad \text{Eq. 178.}$$

From the definition of a vector product (Eq. 27), it follows that the centrifugal force acts along the axis of X in a negative direction; that is, toward the center.

The student should satisfy himself that the same result would have been obtained if the direction Om had been chosen for the axis of Z, or if r had been drawn in the negative direction along the X-axis.

Analogue.

186. Let us now consider a rotating body whose constant angular momentum $I\Omega$ lies at any particular instant along the Z-axis, and at the same time is being rotated about the Y-axis with a constant angular velocity ω. Here, by Eq. 109, the torque is equal to moment of inertia of the wheel multiplied by its angular acceleration; and by Eq. 56½ the total angular acceleration is $\Omega\,\omega$.

Accordingly, the torque L required to drive the rigid rotating wheel about the Y-axis at a rate ω radians per second is given by the following equation:

$$L = -[I\Omega.\omega], \text{ about the } X\text{-axis.} \quad \text{Eq. 179.}$$

From the definition of a vector product, it follows that the torque L acts about the X-axis in a counter-clockwise direction as one looks in the direction from O to X, Fig. 81.

If the bicycle wheel there represented is placed with its hub at a distance r from the origin, it will be necessary to lift the hub with a force $\frac{I\,\Omega.\,\omega}{r}$, in order that the whole system may "precess" (for this motion about the axis of Y is called "precession") at the rate ω.

No single force applied to this system at its center of gravity will make it "precess"; and since a force applied at any other point may be replaced by an equal parallel force at the center of gravity, and a couple, it is clear that the torque $I\Omega\omega$ is a couple, — generally known as the "precessional couple." The solution of Problem 15 is now complete, being quantitatively described in Eq. 179.

EXERCISES.

1. Distinguish carefully between a *centrifugal* and a *precessional* couple.

2. See Young's *General Astronomy*, or some other source, and report to the class upon the precessional motion of the earth.

3. A locomotive is rounding a curve of 200 meters radius at a speed of 60 kilometers an hour; its four driving wheels are each 2 meters in diameter, and have each a moment of inertia 10^{10} C.G.S. units. Find the precessional couple. Does this couple tend to lift the locomotive off the inner or off the outer rail?

4. The armature of a dynamo on a ship weighs 500 kilos. Its axis is at right angles to the length of the ship, and the radius of gyration is 50 cm. If the armature is running at a speed of 500 R.P.M., and the distance between the centers of the bearings is 50 cm., what is the force on the bearings when the ship is rolling at the rate of $\frac{1}{4}$ of a radian per second? Duff, *Mechanics*, page 184.

5. A torpedo boat with propeller making 270 revolutions per minute made a complete turn in 84 seconds. The moment of inertia of the propeller was found, by dismounting it and observing the time of a small oscillation, under gravity, about a horizontal and eccentric axis, to be almost exactly 1 ton-foot^2. Required the precessional torque on the propeller shaft. Suppose the propeller is rotating counter-clockwise as viewed from the bow of the boat, will the precessional couple tend to elevate or depress the stern? Worthington, *Dynamics*, page 158.

6. A gyroscope is set spinning at the rate of 480 R.P.M. Its moment of inertia about the axis of spin is 4800 C.G.S.-units. What weight hung at the end of the axis of spin, i.e., 30 cm. from the pivot which supports the gyroscope, will be required to make the wheel precess at the rate of 3 radians per second?

7. A man riding his bicycle along a level road turns his handle bars so as to steer his wheel to the left. Will this action of his introduce any precessional couple? If so, will this couple tend to right or to upset his wheel?

Digression on Inertia Skeletons and Products of Inertia.

187. The following method of computing products of inertia is along the lines laid down by Worthington, *Dynamics,* pages 64–66 (ed. 1897). This section properly falls under the head of "Rotational Inertia," but could not be discussed in that chapter because centrifugal couples had not then been discussed. The *first step* involves a clear grasp of æquimomental systems and inertia skeletons.

Definition of Equimomental Systems. Any two rigid bodies having equal masses and having equal moments of inertia about the three *principal* axes are said to be *equimomental* because they then have equal moments of inertia about *all* corresponding axes.

This fact will be evident from Eq. 99, § 99, namely,

$$I = A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma. \qquad \text{Eq. 99.}$$

Definition of Inertia Skeleton. For any rigid body it is possible to construct an equimomental body by putting together three uniform rods in such a manner that they bisect each other at right angles and at its center of mass, and in such a manner that the rods coincide in direction with the principal axes of the given body. Such a set of rods is called an *inertia skeleton.* Let us denote by A, B, and

C respectively the principal moments of inertia of the given body, and then proceed to determine for the different skeleton rods the moments of inertia (about perpendicular axes through the center), which we may call A', B', C' respectively, in terms of the given quantities A, B, and C. Let X, Y, and Z (Fig. 83) be the directions of the principal axes of the given body, and, therefore, the directions of the skeleton rods.

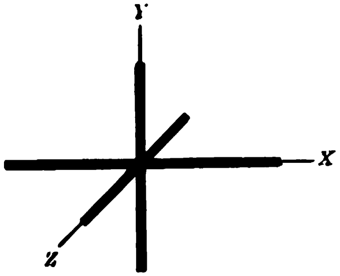


Fig. 83.

From the figure it is clear that the moments of inertia of the skeleton about the axes of X, Y, and Z respectively are

$$B' + C', \quad C' + A', \quad A' + B'.$$

And since the two bodies are equimomental, we have

$$\left.\begin{aligned} B' + C' &= A, \\ C' + A' &= B, \\ A' + B' &= C. \end{aligned}\right\} \qquad \text{Eq. 180.}$$

Having here three equations and three unknown quantities, we solve for the unknown quantities as follows:

$$\left.\begin{aligned} A' &= \tfrac{1}{2}(B + C - A), \\ B' &= \tfrac{1}{2}(C + A - B), \\ C' &= \tfrac{1}{2}(A + B - C). \end{aligned}\right\} \qquad \text{Eq. 181.}$$

If we denote by λ the uniform line density of the rods from which the skeleton is made, and if we consider this quantity as one of the data of the problem, we may compute the required half-lengths (l_1, l_2, and l_3) of the rods as follows:

$$\left.\begin{aligned} A' &= \frac{m_1 l_1^2}{3} = \frac{\lambda}{3} l_1^3, \\ B' &= \frac{m_2 l_2^2}{3} = \frac{\lambda}{3} l_2^3, \\ C' &= \frac{m_3 l_3^2}{3} = \frac{\lambda}{3} l_3^3, \end{aligned}\right\} \qquad \text{Eq. 182.}$$

where m_1, m_2, and m_3 are the respective masses of the rods. Thus, to determine the length of the X-rod, we have, from 181 and 182,

$$\frac{\lambda l_1^3}{3} = \tfrac{1}{2}(B + C - A),$$

in which all the quantities except l_1 are known.

The *second step,* in the computation of products of inertia, is to prove that equimomental bodies, such as we have just been considering, are always acted upon by the same centrifugal couples whenever the rotation in the two bodies is about corresponding axes, and is at the same rate. In other words, if two equimomental bodies are rotating about parallel axes with the same angular speed, each will be acted upon by the same centrifugal couple.

Let A, B, and C denote the principal moments of inertia for *each* of the two bodies; and let D, E, and F with approximate subscripts denote the products of inertia of the bodies. Then, by Eq. 97, § 98, the moment of inertia about an axis whose direction is defined by α, β, γ, will be

$$I = A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma - 2D \cos \beta \cos \gamma - 2E \cos \gamma \cos \alpha - 2F \cos \alpha \cos \beta. \qquad \text{Eq. 97.}$$

Suppose now that we compute the moment of inertia about a line which lies in the XY-plane. Then $\gamma = 90°$, and

$\cos \gamma = 0$. Accordingly, if we indicate the two bodies by subscripts,

$$I_1 = A \cos^2 \alpha + B \cos^2 \gamma - 2\,F_1 \cos \alpha \cos \beta,$$

and

$$I_2 = A \cos^2 \alpha + B \cos^2 \gamma - 2\,F_2 \cos \alpha \cos \beta.$$

Since the bodies are equimomental, however, $I_1 = I_2$. Hence $F_1 = F_2$.

By definition of F, Eq. 96, this becomes

$$\int_0^{M_1} x_1 y_1\, dM = \int_0^{M_2} x_2 y_2\, dM.$$

Now let the angular speeds about the axis of Y be ω_1 and ω_2; these are, by hypothesis, equal, and hence we have the centrifugal couples, L_{1z} and L_{2z} (each about the Z-axis), equal, a result which may be expressed algebraically as follows:

$$L_{1z} = \omega_1^2 \int_0^{M_1} x_1 y_1\, dM_1 = \omega_2^2 \int_0^{M_2} x_2 y_2\, dM_2 = L_{2z} = \omega^2 F.$$

If the rotation had been about the axis of X, the same would have been true. Centrifugal couples of this type we have already studied in § 157.

We now proceed to the *third step*, which is the numerical evaluation of $\int_0^M xy\, dM$ which we have called F. Let the body be represented by its inertia skeleton as shown in Fig. 84. We have chosen (above) the axis of rotation as the axis of Y. We desire to find the product of inertia for the XY-plane. We shall confine our discussion to a solid of revolution; for in this case two rods, say A and B of the inertia skeleton, will be equal. We may, therefore, without further limitation on the generality of the problem, place one of these two rods, say B, along the Z-axis. Accordingly, when the body rotates about the Y-axis, this rod B will not exert any centrifugal couple — since it is at right angles to Y. In computing the integral we shall therefore have need to consider only the rods A and C.

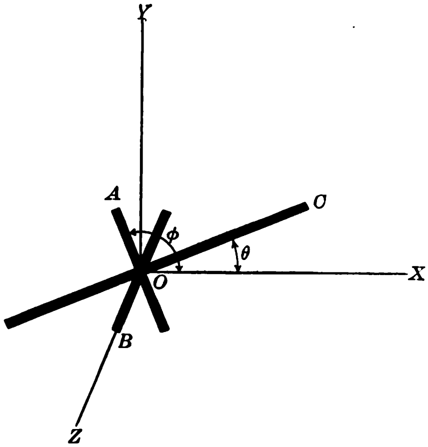


Fig. 84.

Let OC make the angle θ with the axis of X, and OA the angle ϕ with the same axis; then, if r be the distance of the particle dM on either of these rods, from the origin O, we shall have,

$$\left.\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \end{aligned}\right\} \text{ for the rod } C,$$

and

$$\left.\begin{aligned} x &= r\cos\phi, \\ y &= r\sin\phi, \end{aligned}\right\} \text{ for the rod } A.$$

Hence,

$$\int_0^M xy\,dM = \sin\theta\cos\theta\int_0^{M_3} r^2\,dM + \sin\phi\cos\phi\int_0^{M_1} r^2\,dM$$

$$= \sin\theta\cos\theta C' + \sin\phi\cos\phi A'$$

when A' and C' have the same meaning as above, namely, the moments of inertia of the individual rods.

[illegible] how may [illegible]

[illegible] *load on each* and pro-

[illegible] absorbed in rotating a
[illegible] 00 R.P.M. when the
[illegible] coefficient of sliding friction

[illegible] runs at constant
[illegible] it must be conducted

[illegible] down a plane
[illegible] Find the acceleration
[illegible] to the friction of the
[illegible] translational equa-

[illegible] equation

where A is the angular acceleration, and I the moment of inertia. Worthington, *Dynamics of Rotation.*

5. Find the dimensions of the inertia skeleton corresponding to the elliptical disk $\frac{x^2}{4} + \frac{y^2}{9}, = 1$ where the surface density of the disk is 5, and the linear density of the wire in the skeleton, 13.

6. A box is standing on a horizontal plank. The coefficient of friction between them is 0.3. The plank is 12 feet long. Write the equation which determines how high above the ground one can lift either end of the plank without the box beginning to slide.

7. A particle is moving along a straight path with a constant momentum of 25 C.G.S. units. What force, in dynes, acting at right angles to the direction of the momentum, will be required to change the direction of this motion at the rate of 5 radians per second?

8. An inertia skeleton is made up of three rods. The moments of inertia of these rods (taken about an axis perpendicular to the rod at its middle point) are 8, 20, and 30 respectively. Find the principal moments of inertia of the body which is represented by this skeleton.

9. Find the moment of inertia of a fly-wheel which stores up 25 million ergs of energy when its speed is increased from 30 to 90 R.P.M.

10. Find the centrifugal couple acting upon a circular disk 8 cm. in radius, weighing 40 grams, and spinning about an axis, through the center inclined at 45 degrees to the normal, at the rate of $\frac{180}{\pi}$ R.P.M.

11. Prove that the moments of inertia of two similar figures about corresponding axes are in the ratio of the fifth power of their linear dimensions.

12. A particle at the end of a massless string 14 feet long, rotates in a horizontal circle with a period of one-tenth of a second. Its kinetic energy is 400 foot-pounds. Find its mass in engineer's units.

13. A 20-horse-power steam engine is running at the rate of 80 R.P.M. What amount of work can it perform in one revolution?

14. A canvas strap is laid over the fly-wheel (5 feet in diameter) of an engine in order to measure its power. To one end of the strap is attached a weight of 500 lbs.; the other end is held fixed by a spring balance. When the fly-wheel is at rest, the balance reads 500 lbs. When the engine is running at 120 R.P.M., the balance reads 300 lbs. Find the (brake) horse-power of the engine.

CHAPTER V.

DYNAMICS OF ELASTIC BODIES.

188. Up to this point we have been treating all bodies as if they were perfectly rigid, and subject to no changes of either shape or size. For very many purposes this assumption of perfect rigidity is quite allowable. Thus, in ordinary mill-shafting, the pulley and the shaft which carries it may each be treated as absolutely rigid bodies when computing the proper diameter for the pulley or its moment of inertia. But, when one wishes to compute the diameter of shaft necessary to transmit a given amount of power, he must not only understand that a steel shaft is twisted by any torque, but he must know *how much* it is twisted by a given torque.

A drawbridge may be treated as a rigid body by the man whose duty it is to open and close it; but for the engineer who selects the proper dimensions for its members, each piece of iron in it must be regarded as an elastic body which yields to a certain definite extent under each load which passes over the bridge. The *science of elasticity* is, then, one which gives us a *second approximation, so to speak,* to nature. For, as we have already seen (page 1), Dynamics is merely an attempt to draw a picture of the physical universe in the briefest and most accurate terms. The dynamicist is not searching after "causes," but merely seeking to describe the orderly succession of events described in nature.

The fundamental assumption upon which the theory of elasticity is built, is that each part and particle of any given body, though separated from the rest of the body in thought only, may be treated as a separate body. That this is allowable, is not self-evident, but is justified by experience; effects logically computed upon this basis are found to correspond with observed facts.

No new laws of dynamics are introduced or needed for the treatment of this subject; on the contrary, we merely apply the general principles which we have already studied to bodies which suffer change of shape or size when forces act upon them. The science of elasticity is, in short, the dynamics of deformable bodies.

Let us consider any small element of a body: it will consist of a certain volume of matter bounded by a definite closed surface. In general, this surface will be imagined, and will exist only in our own minds.

ROUGH DEFINITION OF STRESS.

189. This element of the body will be acted upon by two kinds of forces: (i) those forces which act upon the entire mass of the element, — forces which, in any particular case, are proportional to the volume of the body. An example of such forces is gravitation. For want of a better name, we shall call these *body forces.* (ii) Those forces which are due to those other parts of the body, which are in immediate contact with the element in question,— forces which act across the bounding surface of the element, and are, other things being equal, proportional to the areas of the surfaces across which they act. These forces are called *stresses.* In the discussion which follows, we shall neglect the body forces and consider only the effects of the stresses.

ROUGH DEFINITION OF STRAIN.

190. Whenever external forces act upon any portion of a body, and change it, either in shape or in size, that portion of the body is said to be *strained*. These changes, whether of dimension or of form, are called *strains*. Stresses are always called into play by these strains, and are such that they tend to restore the body to its unstrained condition.

DISTINCTION BETWEEN SOLIDS AND FLUIDS.

191. Careful examination shows that after all there are only two kinds of changes which may occur in the configuration of elastic bodies:

1. Change of Size; i.e., Compression or Dilation. } Strains.
2. Change of Shape; i.e., Distortion.

In other words, the size of a body may be strained, and the shape of a body may be strained. And since all known bodies are subject to one or both of these changes, elasticity is said to be a *general* property of matter.

But, while all bodies may be changed as indicated, they behave very differently *during* the change. Take, for instance, a glass tube nearly filled with water. Force will be required either to bend or to compress the tube; but the force required to bend the tube is just the same whether the tube be empty, or filled with water. In fact, water has no tendency to recover its shape, but simply adapts itself to any vessel which contains it.

In the case of compression, however, the behavior is quite different. Water, mercury, air, and, indeed, all substances, offer marked resistance to any change of volume.

It is upon this difference of character that the definitions of solid and fluid are based:

(i) Bodies which exhibit both elasticity of shape and of size are called *solids.*

(ii) Bodies which possess elasticity of size but not of shape are called *fluids.*

It should be noted that this distinction between fluids and solids is not a distinction between different substances, but between different bodies. For the same substance may, under different circumstances, assume each of these two states — one needs only to recall the case of ice and water. People who make public demonstrations of liquid air are fond of freezing some mercury about the end of a wooden handle, thus making a quicksilver hammer with which they drive a nail.

DISTINCTION BETWEEN A SOFT SOLID AND A VISCOUS FLUID.

192. If a constant stress, when exerted upon a body, produces a strain which increases continually with the time, the body is said to be *viscous.* If, however, this change of shape is produced only when the stress exceeds a certain definite amount, then the body is a solid. But if the stress, however minute, when continued for a sufficiently long period, produces a continually increasing distortion, then the body must be classed among the fluids, and is called a *viscous fluid.*

A tallow candle is an excellent illustration of a soft solid; while shoemaker's wax, which is so brittle as to be easily broken with a hammer, but which flows under the pressure of its own weight, typifies the viscous fluid.

DISTINCTION BETWEEN GASES AND LIQUIDS.

193. Fluids differ very widely in the manner in which they exhibit their characteristic property; namely, elasticity of size.

(i) " There are certain fluids any portion of which, however small, is capable of expanding indefinitely so as to fill any vessel, however large. These are called *gases.*"

(ii) " There are certain other fluids a small portion of which when placed in a large vessel does not expand so as to fill the vessel uniformly, but remains collected in a mass at the bottom, even when the pressure is removed. These are called *liquids.*" Maxwell, art. "Constitution of Bodies," *Ency. Brit.*

194. We may summarize the preceding classification as follows:

Bodies —
- *Rigid* — not known in nature.
- *Elastic* —
  - *Solid*
  - *Fluid* —
    - *Gases*
    - *Liquids*

TYPICAL BEHAVIOR OF A DUCTILE METAL WHEN STRETCHED.

195. A piece of good wrought iron may be taken as a type of elastic bodies. If a rod of wrought iron be clamped at each end and then stretched, it behaves as shown in the accompanying diagram, Fig. 85.

The essential features here are:

(i) So long as the stress is less than a certain rather definite amount, the strain is proportional to the stress. In other words, the curve connecting stress and [illegible] moderate values, a straight line. The poi[illegible]

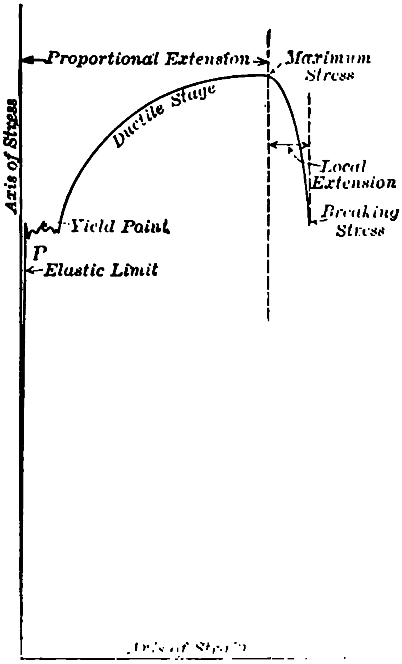


Fig. 85.

at whi[illegible] elastic tionall[illegible] with the [illegible] ality. [illegible] between [illegible] small. [illegible] crystals [illegible] ears when the [illegible] tic. Clay or [illegible] ears when the [illegible] Copper, soft [illegible] amples of ductile [illegible]

[illegible] that part of the occurs is called the *elastic limit* of [illegible] and the elastic Whether the proportionality poi[illegible] considerable impor- the elastic limit depends partly [illegible] If one suspends partly upon what one considers a [illegible] non-magnetic body the presence or absence of which [illegible] ed, he can, with a the "elastic limit." [illegible] gular position of the

[illegible] If the stress is increased [illegible] beyond the proportionality [illegible] reached where the elongation [illegible] be given a twist.

which is well within the limit of elasticity, it will be observed that the suspended body does not at once come back to its initial position, but returns only after the lapse of some considerable time.

For any given substance, the length of delay in reaching the original zero depends principally upon (i) the amount of twist, and (ii) upon the length of time for which the twist was maintained. This behavior of a glass fiber is shown in

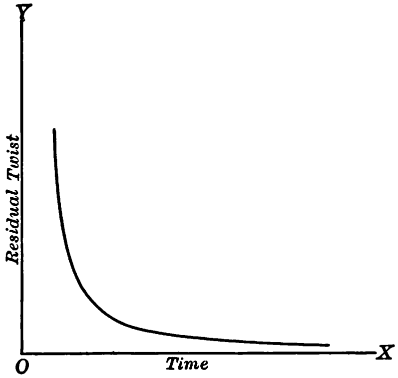


Fig. 87.

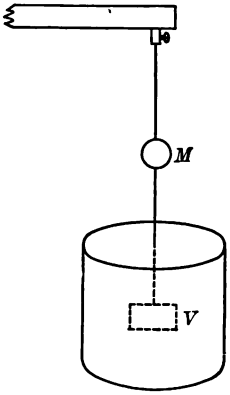


Fig. 88.

Fig. 87. The ordinates here represent the residual strain in the fiber at a time represented by the corresponding abscissae. The effect is most marked in bodies which are non-homogeneous.

In short, twisted fibers behave very much as they would do if there were no " elastic after-effect " in the fibers, but instead there was suspended from the lower end of the fiber a vane dipping into a very viscous liquid. It is evident that a small, quick twist given to the mirror, *M*, will have little effect in displacing the vane, *V*, if the liquid of Breaking Stress) serve to characterize [illegible] the behavior of any ductile substance.

In commercial work it is customary [illegible] limit with the proportionality point, and [illegible] yield point.

BRITTLENESS.

197. When the amount of strain [illegible] the elastic limit and the breaking p[illegible] the material is said to be *brittle*. [illegible] come under this head.

PLASTICITY.

198. If practically none of the st[illegible] stress is removed, the body is said [illegible] putty illustrates this type of substance.

DUCTILITY

199. If only a small part of the [illegible] stress is removed, the body is said [illegible] wrought iron, and gold are well [illegible] metals.

ELASTIC AFTER-STRAIN.

200. Let us now continue [illegible] curve, Fig. 85, which lies between [illegible] limit. There is a time phenomenon [illegible] tance which here makes its appearance [illegible] at the end of a long glass fi[illegible] upon which a small mirror has [illegible] telescope and scale, easily [illegible] suspended body.

If now the lower end of the [illegible]

is set
dies down
has been
elastic after-
se, is to be
after-effect
couple which
position. It
here applied
employed in
term "viscosity
Kelvin and
of liquids:"
upon the *rate*
page 104; Max-
es." Kelvin, art.
et, involving the
quartz, mica, etc.,
therefore, limit
simpler cases which
physics.

I. Simple Extension.

203. The simplest strain which one can imagine is, perhaps, the following; namely, one in which all lines parallel to a given direction are extended, and all lines perpendicular to this direction are unaltered in length.

Let (x, y, z) denote the coördinates of a particle before the strain, and (x', y', z') the coördinates of the same particle after the strain. If the direction of the given extension be chosen for the axis of X, then a simple extension is defined by the following equations:

$$\left.\begin{aligned} x' &= (1 + a_{11})\, x, \\ y' &= y, \\ z' &= z. \end{aligned}\right\} \qquad \text{Eq. 184.}$$

Simple extensions do not occur in nature; on the contrary, it is found that when any body (as, for instance, a piece of wire) is elongated by a pull, it is also contracted laterally. A body which is elongated by rise of temperature behaves differently; but this is not a phenomenon of elasticity.

The meaning of the coefficient a_{11} is evident from Eq. 184. It is the ratio of the increment of length of a line to the original length of the line; and it is called the *extension* of the line. It is evident that the coefficient a_{11} may also be negative, in which case it measures the *contraction* of the line.

II. Simple Shear.

204. Consider a body in which there is some one plane fixed in such a way that not only do all points in that plane remain in that plane, but they retain, during the strain, their original positions in that plane.

Further, let us suppose that all points in any parallel plane remain in their plane, but receive a displacement which is parallel to a given direction in the first plane and which is proportional to the distance apart of the two parallel planes. One can simulate this kind of strain by use of a pack of cards, the bottom one of which remains fixed on the table, while each succeeding one of those on top is shifted, say, $\frac{1}{100}$ millimeter past its neighbor always in the same direction. See Fig. 89.

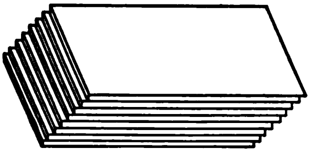


Fig. 89.

Suppose the direction of displacement is chosen as the direction of the axis of X; and suppose that the set of planes which are shifted are those defined by y = constant; then such a strain is described by the following equations:

$$\begin{aligned} x' &= x + a_{12}\,y, \\ y' &= y, \\ z' &= z. \end{aligned} \qquad \text{Eq. 185.}$$

A strain of this kind is called a *simple shear;* and the coefficient a_{12} (which is the ratio of the amount of shift, AB, to the distance AO from the fixed plane, $y = 0$) is called the *amount of the shear.* If the simple shear be very small, it is measured then by the angle which the displacement AB subtends at the point O, Fig. 90. Observe that no change of volume is produced by a simple shear.

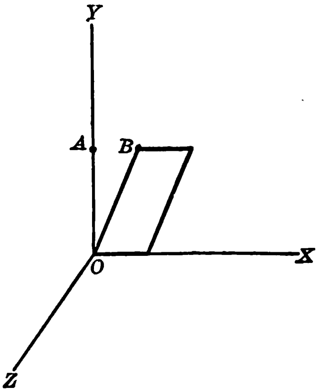


Fig. 90.

III. Homogeneous Strain.

205. Let us now suppose that a body is strained in such a way that all lines lying parallel to axes of X, Y, and Z receive simple extensions a_{11}, a_{22}, and a_{33}, respectively; and let us suppose that, at the same time, each of the two coördinate planes parallel to X receives a shear along the direction of the axis of X; and the same for the pairs of planes parallel to the axes of Y and Z respectively.

The equations which describe such a strain as this are evidently made up of terms such as those which appear in Eqs. 184 and 185 and are as follows:

$$\left.\begin{aligned} x' &= (1 + a_{11})\, x + a_{12}\, y + a_{13} z, \\ y' &= a_{21} x + (1 + a_{22})\, y + a_{23} z, \\ z' &= a_{31} x + a_{32} y + (1 + a_{33})\, z. \end{aligned}\right\} \qquad \text{Eq. 186.}$$

Such a strain is called a *homogeneous strain*, and is the general case of all the strains which we shall study in this course. Coefficients having both subscripts alike are *extensions*; those having different subscripts are *shears*.

From the fact that the equations are linear, it is evident that parallel planes will be transformed into parallel planes, parallel lines into parallel lines, and spheres into ellipsoids. Any such ellipsoid is called a *strain ellipsoid*; and the principal axes of this ellipsoid are called the *principal axes of the strain*. In general, the principal axes of the ellipsoid will not coincide with their original (unstrained) direction; but they can be brought into coincidence by rotating the strained body *as a whole*. A rotation of this kind does not imply any strain in the body, and does not at all affect the elastic problem.

Pure Strain.

When the strain is such that the principal axes have the same direction before and after the deformation, the strain is said to be *pure*. *In general, therefore, a homogeneous strain consists of a pure strain and a rotation.*

Let us now choose the principal axes of the strain ellipsoid for our axes of reference. Then a line of unit length parallel to the axis of X will not have its direction changed; but its length will become, say, $1 + e$; in the same way, a unit length parallel to Y becomes $1 + f$; and one parallel to Z acquires a length of $1 + g$. These quantities e, f, and g are special values of a_{11}, a_{22}, and a_{33} in Eq. 186, namely, the values which the a's assume when the principal axes of strain are chosen as coördinate axes.

IV. Cubical Dilatation.

206. Imagine a unit cube whose edges coincide with the principal axes of the strain. Its volume before the strain is unity. The volume of the parallelopiped which results from the strain is $(1 + e), (1 + f), (1 + g)$. If now the strain be so small that we can neglect the squares and higher powers of the elongations, we may write the volume of the parallelopiped as $1 + e + f + g$.

Hence the change in unit volume due to the strain is $e + f + g$. Accordingly the defining equation for *cubical dilatation*, a quantity which we shall denote by δ, is

$$\delta = e + f + g. \qquad \text{Eq. 187.}$$

If the dilatation is the same in the direction of each of the three axes, it is said to be *uniform*. A uniform dilatation is described algebraically as follows:

$$\left.\begin{array}{l} e = f = g, \\ \delta = 3\,e. \end{array}\right\} \begin{array}{c} \text{Equations of uniform} \\ \text{Dilatation.} \end{array} \qquad \text{Eq. 188.}$$

V. Pure Shear in Terms of Extension and Contraction.

207. Let us now consider still another special case of a homogeneous strain, one which is called a plane strain.

We imagine the rhombus $ABCD$ to represent the section of a solid body, and that *each plane of the body parallel to the plane of the paper receives the same strain.* By making these assumptions, one is able to reduce the three-dimensional problem to a two-dimensional one.

Now let the original body receive an extension, f, along the direction BD (Fig. 91), so that the line BD is extended into the line $B'D'$. At the same time, let the body receive a contraction, e, along the line AC, so that AC is contracted into $A'C'$. It is evident that this strain will be a pure one, since the directions of the principal axes are not changed by the operation; and that the figure passes through a square during the strain.

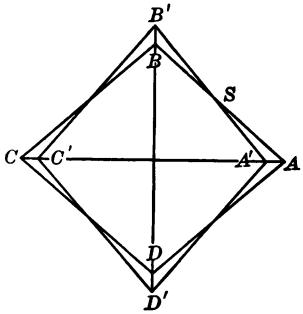


Fig. 91.

If now the *volume of the body is not to be changed during the strain, we must have*

$$(1 + e)(1 + f) = 1. \qquad \text{Eq. 189.}$$

We assume that the third dimension, the one perpendicular to the plane of the paper, remains unchanged; and we assume, as heretofore, that the strains are so small that squares and higher powers may be neglected.

Hence Eq. 189 is equivalent to $e = -f$.

Hence the effect of the strain will be to transform the rhombus $ABCD$ into an *equal* rhombus $A'B'C'D'$. Hence $AD = A'D'$.

To see just what has been done by introducing the extension f, let us rotate and translate the strained rhombus as a whole, in its plane, so that the side $A'D'$ coincides with the side AD. The effect is that indicated in Fig. 92; in other words, the effect is the same as that produced in the leaves of a book when one side of the book's cover is sheared past the other side. The upshot of the whole matter is that *a pure shear is equivalent to an extension in one direction together with an equal contraction at right angles to the extension.*

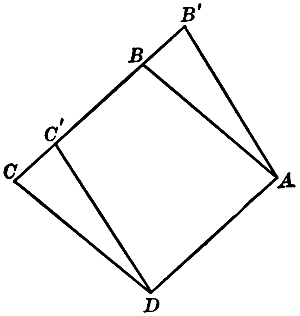


Fig. 92.

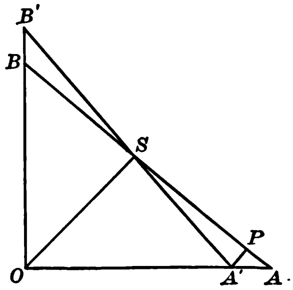


Fig. 93.

It remains now only to find the amount of the shear thus produced, in terms of the extension f.

MEASURE OF SHEAR.

It is clear that the angle of shear BAB' in Fig. 92 is just twice the angle ASA' in Fig. 91.

In Fig. 93 is reproduced one quadrant of Fig. 91, from which it is clear that $\frac{A'P}{A'S}$ is the measure of the angle $A'SP$;

for it must be remembered that all these angles introduced by strain are small.

Hence the angle of shear $= 2\left(\frac{A'P}{A'S}\right) = 2\left(\frac{A'A \sin 45^\circ}{AS}\right)$.

But $A'S = OS$, and hence $A'S = \sqrt{\tfrac{1}{2}}\,\sqrt{OA^2}$.

$$\text{Angle of shear} = 2\,\frac{AA'\sqrt{\tfrac{1}{2}}}{OA\sqrt{\tfrac{1}{2}}} = 2\,\frac{AA'}{OA} = 2\,e.$$

If, therefore, e and $-e$ are the two extensions, the shear is $2\,e$.

VI. General Shear in Terms of Extension and Contraction.

208. In the case immediately preceding, we have assumed that no change of volume was produced by the strain; and, in order to secure this, we found it necessary to make the extension in one direction numerically equal to the contraction in the other, Eq. 189.

Let us now suppose that the material under consideration receives any extension e along one principal axis, and any extension f along the other principal axis. If x and y are the original lengths of two lines parallel to the respective principal axes, and if x' and y' are the strained lengths of these same two lines, then we may write

$$\begin{aligned} x' - x = ex &= \tfrac{1}{2}\,(e + f)\,x + \tfrac{1}{2}\,(c - f)\,x, \\ y' - y = fy &= \tfrac{1}{2}\,(e + f)\,y - \tfrac{1}{2}\,(e - f)\,y. \end{aligned} \qquad \text{Eq. 190.}$$

Considering the extreme right-hand members of these two equations, it is clear that the first terms represent a uniform dilatation $\tfrac{1}{2}\,(e + f)$ along each of the two principal axes; but the last terms of the right-hand member represent an extension $\tfrac{1}{2}\,(e - f)$ along the X-axis, and a contraction $-\tfrac{1}{2}\,(e - f)$ along the Y-axis.

rhombus $ABCD$ into an *equal* rhombus [illegible] $AD = A'D'$.

To see just what has been done by in[illegible] sion f, let us rotate and translate the [illegible] a whole, in its plane, so that the side A'[illegible] side AD. The effect is that indicated [illegible] words, the effect is the same as that p[illegible] of a book when one side of the book's [illegible] the other side. The upshot of the [illegible] *a pure shear is equivalent to an exte*[illegible]

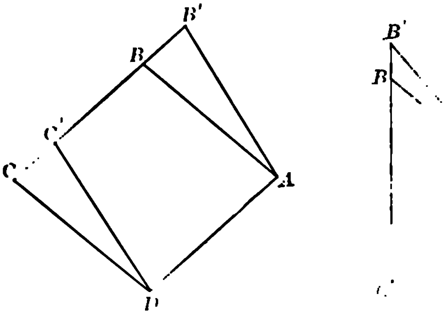


Fig. 92.

[illegible] *equal contraction in* [illegible]

It remains now only to find the [illegible] produced in terms of the extension [illegible]

MEASURE OF SH[illegible]

It is clear that the angle of sh[illegible] [illegible] in Fig. 9[illegible]

[illegible]

[illegible] there is [illegible] to [illegible] paper.

[illegible] are those [illegible] geneous [illegible] in which [illegible] parts [illegible] elonga- [illegible] rtional to [illegible] *part of the*

[illegible] in Fig. 1[illegible]

of a body. Whether these surfaces be real or imaginary is a matter of indifference.

The *intensity of stress* is measured by the ratio of the force acting over any surface to the area of that surface. If F denote the force acting on the surface S, the intensity of the stress is $\frac{F}{S}$. In general, this will represent only the *average* value of the intensity of the stress, and is to be used only when the stress is uniform.

At any one point on a surface the intensity will be accurately given by the differential coefficient, $\frac{dF}{dS}$.

It is customary among scientific men to use simply the word "stress" instead of "intensity of stress." To indicate a force, F, they employ either the word "force" or the expression "total stress," just as "total pressure" is sometimes employed to indicate the "force" which a liquid exerts against the wall of its containing vessel.

It must not be forgotten that the forces which produce stresses are just like other forces; they are measured in the same manner, and are subject to the same conditions of equilibrium, composition, and resolution.

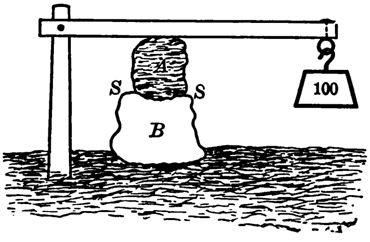


Fig. 94.

Thus, if we consider the surface SS (Fig. 94), separating two bodies A and B, it is evident that the total stress over the surface SS must be in equilibrium with all the other forces acting upon the body A, including the weight of A, as well as the force exerted by the lever and by the pressure of the earth's atmosphere. This stress we might call the reaction of B upon A; and in exactly the same manner, the total stress over the interface SS may be considered as the reaction of A upon B; and must therefore be the resultant of all the other forces acting upon B.

From either point of view, the stress is the same in amount; as, indeed, is demanded by Newton's Third Law.

RESOLUTION OF STRESSES.

211. In general, the stress, P, at any point on a surface will not be either normal or parallel to that surface, but will lie in an oblique direction as indicated in Fig. 95.

Stress, being a vector quantity, may be resolved into a normal and a tangential component as shown.

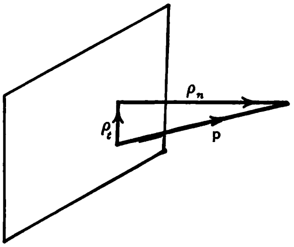


Fig. 95.

If these components be indicated by p_n and p_t respectively, then we shall have the vector equation

$$\bar{p}_n + \bar{p}_t = \bar{P},$$

provided, of course, the components each act upon surfaces of the same area as that acted upon by the resultant $\bar{P}$. A normal stress which tends to separate the bodies in contact at any surface is said to be positive; a stress tending to push the two bodies together is negative. It must be carefully noted that the equations of equilibrium in dynamics apply, in the first instance, to forces; and if, therefore, one wishes to pass from an equation of equilibrium between forces to one of equilibrium between stresses, he must divide each member of the former equation by the same element of area.

I. Simple Longitudinal Stress.

212. A wire which is elongated by a simple pull, or a column which is shortened by a simple push, represents as simple a stress as could well be imagined.

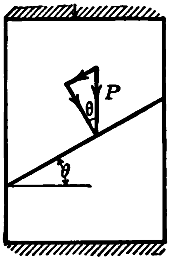


Fig. 96.

Over any cross-section perpendicular to the length of the rod, it is clear that the stress will be entirely normal; and over any plane which is parallel to the rod, the normal stress will be zero. We proceed to find the stress over *any* plane whose normal is inclined at an angle θ *to the axis of the rod.* Let A be the area of any normal cross-section; then A', the area of any oblique section, will be

$$A' = \frac{A}{\cos\theta}.$$

The normal component of the stretching force F will be f_n where

$$f_n = F\cos\theta;$$

and hence, if P denote the stress along the axis of the column, the *component of stress normal to the plane*

$$= p_n = \frac{f_n}{A'} = \frac{F}{A}\cos^2\theta = P\cos^2\theta. \qquad \text{Eq. 191.}$$

In like manner, the tangential component of the force f_t is

$$f_t = F\sin\theta,$$

and hence the *tangential component of stress*

$$= p_t = \frac{f_t}{A'} = \frac{F}{A}\sin\theta\cos\theta = P\sin\theta\cos\theta. \qquad \text{Eq. 192.}$$

Evidently p_n is a maximum when θ is zero, and p_t is a maximum when θ is 45°. One might expect the tangential stress to be greatest when measured over a surface parallel to the force F; i.e., when θ is 90°; the reason this is not so is because, under this condition, A' becomes infinitely large compared with A. When $\theta = 90°$, the tangential component of the force is, indeed, a maximum; but it is never to be forgotten that stress is the ratio between a force and the area over which it acts.

If the oblique plane be chosen at right angles to that which we have just been considering, i.e., so that its angle with the normal cross-section is $\theta + 90°$, then

$$p_n = P \sin^2 \theta,$$

and

$$p_t = P \cos \theta \sin \theta.$$

Here again the maximum tangential stress occurs when θ is 45°.

II. Simple Shearing Stress.

213. An interesting special case, and one of great importance in practice, is that in which an element of the body, say a cubical block, such as that shown in Fig. 97, has a pair of tangential forces, Q, applied to two opposite faces. All planes parallel to the face ab will in this case be subject to a tangential stress only.

Such a stress as this is called a *simple shearing stress*. But if the block were subject to this pair of forces only, it is clear that the block, as a whole, would be set into rotation by the couple, Qac, which would be produced by Q alone.

This case, in which only two pairs of shearing stresses are applied, is evidently a case in two dimensions; for we have assumed that all the stresses are the same in any plane parallel to the plane of the paper.

We shall see presently that shearing *stresses* may be produced in other ways than that mentioned above, just as we saw that shearing *strains* can be produced in more than one way.

The effect of a pair of shearing stresses, R and Q (such as those indicated in Fig. 98), over any plane in the block, is to be computed in the same manner as for ordinary forces, except that areas have to be taken into account. Thus, one may inquire what the normal stress, P, will be over the diagonal plane, *ad*. Let A denote the area of one face of the cube; and let us assume that $R = Q$ so that there will be no rotation of the cube.

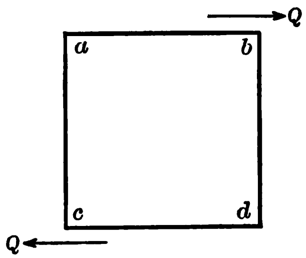


Fig. 97.

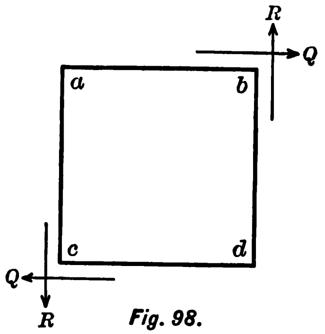


Fig. 98.

Area of diagonal plane $= A\sqrt{2}.$
Denote normal stress over diagonal plane by P, then
Force acting normal to diagonal plane $= A\sqrt{2} \cdot P.$
Force over face $cd = QA =$ force over face ac.

Resolving each of these forces perpendicular to *ad*, and adding, we have

$$QA\sqrt{\tfrac{1}{2}} + QA\sqrt{\tfrac{1}{2}} = PA\sqrt{2}.$$

Hence

$$P = Q.$$

In other words, the normal stress over the *diagonal* of the cube is exactly equal to the shearing stress over the *face* of the cube; while the shearing stress over the diagonal becomes zero.

III. Most General Stress at a Point.

214. Let us now consider an element (in a body) each face of which is subjected, in general, to a normal stress and a shearing stress.

Let us suppose the stresses at any point, O, to be in equilibrium, and to be given for each of the three coördinate planes, as follows:

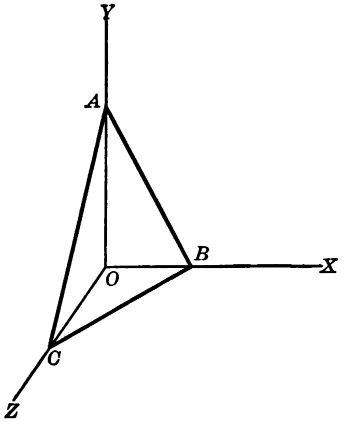


Fig. 99.

X_x = normal stress over YZ-plane; i.e., over the plane to which the axis of X is normal.

X_y = tangential stress over XZ-plane resolved in direction of X.

X_z = tangential stress on XY-plane resolved in direction of X.

Y_x = tangential stress over YZ-plane resolved in direction of Y.

Y_y = normal stress over XZ-plane.

Y_z, Z_x, Z_y, Z_z, are defined in an analogous manner, the capital letter indicating the direction of the stress, the subscript indicating the direction of the normal to the surface, over which the stress acts. When these nine components of stress are given, we shall see that it is possible to obtain the components of stress over any other surface passing through the point O. Let ABC (Fig. 99) be such a surface. The plane ABC, together with the three coördinate planes which intersect at O, will form a tetrahedron, over three of whose faces the stresses are completely known. The problem is to find the stress over the remaining face, ABC. To do this we have merely to write the equation of equilibrium for each of the three axes X, Y, Z. By taking the tetrahedron sufficiently small, we can make the plane ABC pass as nearly through the point O as we may wish.

Let D be the area of the triangle ABC, and let (l, m, n) be the direction cosines of its normal.

Then the areas of the other faces of the tetrahedron will be

$$\left.\begin{aligned} AOC &= lD, \\ BOC &= mD, \\ BOA &= nD. \end{aligned}\right\} \qquad \text{Eq. 193.}$$

We may now write at once the equations of equilibrium between all the forces acting upon the tetrahedron.

Let F, G, and H be the components of the stress over the face D due to all the matter outside the tetrahedron. Then equating forces (not stresses) along the X-axis, we have

$$FD = X_x lD + X_y mD + X_z nD,$$

or

$$\text{and in like manner} \quad \left.\begin{aligned} F &= lX_x + mX_y + nX_z, \\ G &= lY_x + mY_y + nY_z, \\ H &= lZ_x + mZ_y + nZ_z. \end{aligned}\right\} \qquad \text{Eq. 194.}$$

These equations describe completely and unambiguously the amount and direction of the stress over any plane ABC passing through the point O. If the resultant stress over this plane be indicated by N (not normal to the surface), it may be obtained from the following equation, since N, F, G, and H each act over the same area D:

$$N^2 = F^2 + G^2 + H^2. \qquad \text{Eq. 195.}$$

The student who pursues the subject of elasticity farther will discover a great variety of ways for expressing this same general result.

So far we have considered only the translational forces acting upon the tetrahedron. But if this element is in equilibrium, as we have supposed, we must also have the rotational forces (moments of force) in equilibrium, as in the case of the simple shear (Fig. 98).

Consider any minute cube of the pyramid (Fig. 99), and let this cube be represented by Fig. 100. The forces tending to rotate this cube about the axis of X are $Y_z\,dy\,dx$ and $Z_y\,dz\,dx$. The arms through which these two forces act are

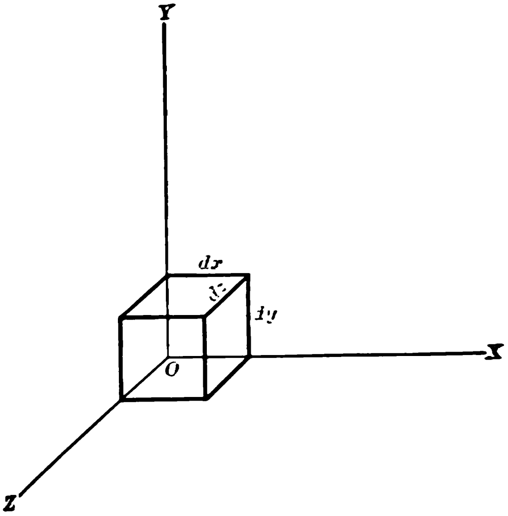


Fig. 100.

dz and dy respectively. The torque due to the first force is therefore $- Y_z\, dy\, dx\, dz$; and the torque due to the second force is $+ Z_y\, dx\, dz\, dy$. The condition of no rotation (angular equilibrium) is that the sum of these torques shall be zero.

This condition is expressed algebraically as follows:

$$\left.\begin{aligned} Y_z - Z_y &= 0, \\ Z_x - X_z &= 0, \\ X_y - Y_x &= 0. \end{aligned}\right\} \qquad \text{Eq. 196.}$$

If now we change our notation in the manner indicated by the following equations,

$$\left.\begin{aligned} &X_x = P, \quad Y_y = Q, \quad Z_z = R, \\ &Y_z = Z_y = S, \quad Z_x - X_z = T, \quad X_y = Y_x = U, \end{aligned}\right\} \text{Eq. 197.}$$

we shall have our general equations in their final form, namely,

$$\left.\begin{aligned} F &= lP + mU + nT, \\ G &= lU + mQ + nS, \\ H &= lT + mS + nR. \end{aligned}\right\} \qquad \text{Eq. 198.}$$

The quantities P, Q, R, S, T, U, are called *the six components of the stress* at any point.

The student should see to it, at once, that the physical meaning of each of these twelve symbols and of each of these three equations is perfectly clear.

IV. Fluid Pressure.

215. An interesting special case of Eq. 198 is that met with in the case of fluids which, by definition, cannot maintain any shearing stress. Fluids, it will be remembered, possess elasticity of volume, but no elasticity of shape. See § 191.

Another, and, for present purposes, perhaps better way of stating the case, is to say that, at any point, in a fluid at rest, the stress over any surface is entirely normal to that surface. It is only when fluids are in motion that they exhibit the temporary shearing stress which we call "viscosity."

Let us now describe this property of fluids at rest by writing

$$S = T = U = 0.$$

Hence our general equations become

$$\left.\begin{aligned} F &= lP, \\ G &= mQ, \\ H &= nR. \end{aligned}\right\} \qquad \text{Eq. 199.}$$

Since now F, G, and H are merely the X, Y, and Z-components of the *total* stress, N, over the given surface (say, the surface D of Fig. 99), and since the stress, in this case, is normal by definition, we may write, in general, also,

$$\left.\begin{aligned} F &= lN, \\ G &= mN, \\ H &= nN, \end{aligned}\right\} \qquad \text{Eq. 200.}$$

and hence

$$P = Q = R = N, \qquad \text{Eq. 201.}$$

which shows that at any point in a fluid at rest the pressure is the same in all directions; for the four quantities which appear in Eq. 201 are simply the normal pressures on the four faces of an elementary tetrahedron such as that described in Fig. 99. The *average* pressure over any element, whether solid or liquid, is

$$\tfrac{1}{3}(P + Q + R).$$

V. A Pure Shearing Stress in Terms of a Simple Push and Pull.

216. Let us now consider a two-dimensional case in which we have only a positive stress, P, in one direction, and an equal negative stress, Q, in a direction at right angles to P, while in a direction perpendicular to each of these two there is no stress whatever. Let AB be the trace of any plane perpendicular to the plane of the figure, and whose normal is inclined at an angle θ to the direction of the stress, P. We now proceed to inquire what the stress over this plane AB is.

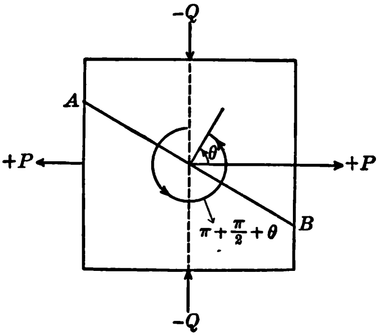


Fig. 101.

From Case I, Eq. 191, we have, remembering that $Q = -P$,

$$\text{Normal Component of } P = P\cos^2\theta,$$

$$\text{Normal Component of } Q = -P\cos^2\left(\theta + \frac{\pi}{2} + \pi\right).$$

The angle to be employed here in each of these stresses is that which lies between the direction of the stress and the normal, the angle in each case to be measured in the same sense (here clockwise) and to the normal drawn on the same side of the surface AB, and hence

$$\text{Total Normal Stress over } A\,B = P\,(\cos^2\theta - \sin^2\theta). \qquad \text{Eq. 202.}$$

In like manner from Eq. 192 we have

Tangential Component of $P = P \sin \theta \cos \theta$,

Tangential Component of Q

$$= - P \sin\left(\theta + \frac{3\pi}{2}\right) \cos\left(\theta + \frac{3\pi}{2}\right),$$

and hence

Total Tangential Stress over $AB = 2\,P \sin \theta \cos \theta$. Eq. 203.

From Eq. 202 it will be seen at once that the normal stress over AB becomes zero when $\theta = 45°$; while for this same value of θ, the tangential stress becomes a maximum, namely, P. The result is then that a simple shearing stress over any plane AB may be replaced by two simple longitudinal stresses at right angles to each other, and inclined at 45° to the plane AB.

VI. General Shearing Stress in Terms of Push and Pull.

217. In the case immediately preceding we have assumed the two longitudinal stresses, P and Q, to be equal in amount. Let us now suppose these two stresses to have quite different values. Then adding the normal components of the stresses, P and Q, we have the

Total Normal Stress over $AB = P \cos^2 \theta + Q \sin^2 \theta$, Eq. 204.

and in like manner

Total Tangential Stress over AB

$$= P \sin \theta \cos \theta - Q \sin \theta \cos \theta = (P - Q) \sin \theta \cos \theta.$$

Eq. 205.

Consequently a pair of stresses, made up of a push and a pull at right angles to each other, are equivalent, over any surface defined by θ, to a tangential stress defined by 205 plus a normal stress defined by 204.

VII. Heterogeneous Stress.

218. When we consider the general equations of stress (198), it is important to remember that these were derived merely for the purpose of computing the stress over *any* plane passing through a certain given point. For this particular point, the six stresses P, Q, R, S, T, U, are constants; the only variables are l, m, n, which determine the aspect of the plane. But if, in the case of any stress, the components P, Q, R, S, T, U, remain constant throughout the entire body, the stress is said to be *homogeneous*. If, however, P, Q, R, S, T, U, vary from point to point in the body, the stress is said to be *heterogeneous*.

Relations between Stresses and Strains. Elastic Moduli.

219. In the foregoing pages, we have discussed questions of stress and questions of strain as two analogous, but entirely independent, subjects. The actual problems which one encounters in the science of elasticity are, however, generally of the following type : " Given certain stresses in a certain body; find the strain produced." Or, conversely, " Given certain strains in a certain body; find the stress necessary to produce them." To the engineer and to the physicist, it is, therefore, a matter of great importance to discover the relations which connect stresses and strains, and to express these in an algebraic way so that they may be used in computation.

Hooke's Law.

220. The first important step in this direction was made by Robert Hooke in 1660, and published by him in 1676 under the form of the following cryptogram.

ceiiinosssttuu.

In 1678 he rearranged and translated these letters as follows: " *Ut tensio sic vis;* that is, the Power of any spring is in the same proportion with the Tension thereof." This important experimental fact connects the load which is applied to an elastic body with the deformation which is produced in that body. Since the time of Hooke, this law has been generalized into the following statement:

" *The deformation produced is proportional to the load producing it.*" Or, since stress and force are proportional, we may say,

Stress is a linear function of strain.

And this is the form which Hooke's Law actually takes in the theory of elasticity.

Two limitations should be noted: (i) The law applies only to bodies which are strained within the elastic limit. See § 195. (ii) The law applies only to bodies which are strained at constant temperature. For by changing the temperature of a body, we may produce strains in it without introducing any stress whatever.

The best evidence for the truth of Hooke's Law is that first pointed out by Stokes, namely, the fact that all elastic solids can be made to vibrate isochronously; for, equal periods of vibration, at all amplitudes, means that the force acting upon the body is proportional to the displacement.

A still better way of viewing Hooke's Law is, perhaps, the following: If a strain e is produced by a stress P, and a strain e' by a stress P', then, so long as we remain within the elastic limit, the stress $P + P'$ will produce the strain $e + e'$.

Elastic Moduli.

If we were to express this law in algebraic form, the equation would be as follows:

$$\text{Stress} = \text{constant} \times \text{strain}.$$

The " constant " which is thus defined is called a *modulus of elasticity*. In general, a modulus of elasticity is the number which expresses the ratio between the stress and the corresponding strain. Since a strain is always a pure number, it will be at once evident that the dimensions of any modulus are the same as those of the corresponding stress; namely, a force divided by an area.

221. As will be seen by looking over the equations (190) of a homogeneous strain, the most general strain of this type is made up of two simple strains: (i) a uniform expansion or contraction and (ii) a simple shear. The two important moduli are, therefore, those which connect (i) change of volume with fluid pressure, and (ii) the amount of shear with the shearing stress.

I. Bulk Modulus.

222. Let us suppose that the volume of a body is changed without any change of shape; call the dilatation δ. Then, if P be the uniform pressure (or tension) by which this dilatation is produced, the

$$\text{Bulk Modulus} = k = \frac{P}{\delta}. \qquad \text{Eq. 206.}$$

Thus for glass, the bulk modulus, on the C.G.S. system, averages about 4×10^{11}; while for steel, k is about 15×10^{11}.

II. Rigidity Modulus.

223. Let us next suppose that the shape of a body is changed by a simple shearing stress without any change in size. If we denote the strain (i.e., the angle of shear) by θ, and the shearing stress by S, the

$$\textit{Rigidity Modulus} = n = \frac{S}{\theta}. \qquad \text{Eq. 207.}$$

For glass, n is, in round numbers, 2×10^{11}.
For steel, n is, in round numbers, 9×10^{11}.

III. Young's Modulus.

224. The ratio of the longitudinal stress to the longitudinal strain in any body is called *Young's Modulus.* But since we have seen, § 221, that all the elastic constants of isotropic bodies can be derived from the bulk modulus and the rigidity modulus, it is clear that Young's Modulus must be expressible in terms of n and k. Perhaps the most elegant method of obtaining this relation is that employed by Poynting and Thomson, which is as follows:

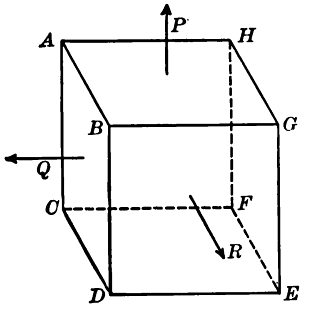


Fig. 102.

Consider the stressed body to be made up of elementary (i.e., very small) parallelopipeds, such as that shown in Fig. 102. If the stress is homogeneous and the body isotropic, as we suppose, each of these little elements will be stressed and strained in the same way.

Let the normal stress over the face A, B, G, H and over the opposite face be P. Let the normal stress over the face

ABCD and over the opposite face be Q. And finally let the face *BDEG* and its opposite be subject to a normal stress R. Suppose now that there are no stresses other than these three acting upon the body. As heretofore, e, f, and g will indicate the extensions in the direction of the stresses P, Q, and R respectively. It is understood, of course, that the entire body, and therefore each of these elementary parallelopipeds, is in equilibrium under the stresses above mentioned.

In order to obtain the general relations between e, f, and g on the one hand, and P, Q, and R on the other, we shall imagine these stresses to act separately, and shall then proceed to add their individual effects.

Let λ be the extension which *unit* stress will produce in its own direction; and let μ be the contraction which *unit* stress will produce in all directions at right angles to the stress. Then, if P acts alone, the three mutually perpendicular extensions are λP, $-\mu P$, and $-\mu P$.

In like manner, when Q acts alone, the extensions are $-\mu Q$, λQ, and $-\mu Q$. So also in the case of R acting alone, the extensions are $-\mu R$, $-\mu R$, and λR.

Let us now return to our original supposition, namely, that all three stresses, P, Q, and R, act simultaneously. Then for the total extension, along the directions of P, Q, and R respectively, we shall have

$$\left.\begin{aligned} e &= \lambda P - \mu Q - \mu R, \\ f &= -\mu P + \lambda Q - \mu R, \\ g &= -\mu P - \mu Q + \lambda R. \end{aligned}\right\} \qquad \text{Eq. 208.}$$

The only difficulty with these equations is that we do not yet know the values of λ and μ.

Accordingly we proceed to determine these in terms of

k and n, the two fundamental moduli which are the data of this discussion. (i) Let us apply to *each* face of the parallelopiped a tension P. This will produce a voluminal strain $\frac{P}{k}$ (Eq. 206), and hence a linear extension $\frac{P}{3k}$. We have supposed $P = Q = R$, and we find $e = f = g = \frac{P}{3k}$. Introducing these values of e, f, and g into any one of equations 208, we obtain

$$\frac{1}{3k} = \lambda - 2\mu. \qquad \text{Eq. 209.}$$

(ii) Let us now apply a tension P to one face of the parallelopiped, and an *equal* compression, $-Q$, to a second face, while the third face remains free from stress, i.e., $R = O$. This will produce a pure shear in the PQ plane; as we have seen (Case II, § 213) the amount of this shearing stress is P; and since the modulus of rigidity is n, the shearing strain will be $\frac{P}{n}$.

And since, § 207, the angle of shear, i.e., the shearing strain, is twice the extension, we may write

$$2e = -2f = \frac{P}{n}.$$

Remembering that $P = -Q$, and $R = O$, insert these values in Eq. 208 and obtain

$$\frac{1}{2n} = \lambda + \mu. \qquad \text{Eq. 210.}$$

Solving for λ and μ from Eqs. 209 and 210, we obtain

$$\left.\begin{aligned} \mu &= \frac{1}{3}\left(\frac{1}{2n} - \frac{1}{3k}\right) = \frac{3k - 2n}{18nk}, \\ \lambda &= \frac{1}{3}\left(\frac{1}{n} + \frac{1}{3k}\right) = \frac{3k + n}{9nk}. \end{aligned}\right\} \qquad \text{Eq. 211.}$$

Let us denote Young's Modulus by M, then its defining equation is

$$P = Me. \qquad \text{Eq. 212.}$$

Young's Modulus comes into play when the body is acted upon by a single longitudinal stress P. To describe this state of affairs, we may write

$$Q = R = O,$$

and our general equations (208) then become

$$\left.\begin{aligned} e &= \lambda P, \\ f &= -\mu P, \\ g &= -\mu P. \end{aligned}\right\}$$

Comparing the first of this trio with Eq. 212, we find

$$M = \frac{1}{\lambda} = \frac{9\,nk}{3\,k + n}\,. \qquad \text{Eq. 213.}$$

We have therefore completely determined Young's Modulus in terms of the bulk modulus and the rigidity modulus.

An excellent geometrical discussion of this same problem is to be found in Ewing's *Strength of Material,* pages 16–20; also in Stewart and Gee's *Practical Physics,* Vol. I, pages 170–175.

IV. Poisson's Ratio.

225. A quantity of great importance in the theory of elasticity is *the ratio of the lateral contraction to the longitudinal extension* in the case just mentioned, namely, where a rod receives a stress parallel to its length. This ratio is called *Poisson's Ratio.* Its defining equation is as follows:

$$\sigma = \frac{f}{\epsilon} \quad \text{when } Q = R = 0, \qquad \text{Eq. 214.}$$

or, in terms of our general equations,

$$\sigma = \frac{\mu}{\lambda}.$$

And hence, by Eqs. 211,

$$\sigma = \frac{3\,k-2\,n}{2\,(3\,k+n}. \qquad \text{Eq. 215.}$$

We have, therefore, determined Poisson's Ratio in terms of the data of the problem n and k.

EXPERIMENTAL DETERMINATION OF ELASTIC CONSTANTS.

226. We have seen that, in the theory of elasticity, the two fundamental constants are the bulk modulus, k, and the rigidity modulus, n. But in the laboratory and in the testing room, one finds that the most simply determined of all these quantities is

Young's Modulus

which, therefore, we shall first consider.

(i) *Case of a Wire.* If the material to be measured is available in the form of a wire, the best method is probably one which, so far as we can learn, was first employed in Kings College, London, namely, suspend two wires from a beam; to the lower end of one attach a rather finely and accurately divided scale; to the other attach a vernier in such a way that the vernier slides smoothly but snugly along the scale. The wire whose modulus is to be determined — one of the two wires just mentioned — carries also a scale-pan, while the other wire carries a constant weight sufficient to keep it stretched straight. (See Watson's *Practical Physics,* Chap. V, or Glazebrook & Shaw, *Practical Physics,* page 187.)

As various loads are successively placed in the scale-pan the wire receives a simple longitudinal stress, and the corresponding elongation can be read directly from the scale and vernier. The diameter of the wire can be determined either with an accurate pair of calipers or by weighing the wire in a specific gravity bottle.

If the wire is sufficiently long a steel tape will give its length with ample accuracy, while the load is easily determined by a pair of balances.

If m be the mass of the load, l the length and d the diameter of the wire, e the elongation, and g the acceleration of gravity, then we have as the laboratory equation for Young's Modulus

$$M = \frac{\text{longitudinal stress}}{\text{longitudinal strain}} = \frac{\dfrac{mg}{\pi \dfrac{d^2}{4}}}{\dfrac{e}{l}} = \frac{4\, mgl}{\pi\, e\, d^2}. \qquad \text{Eq. 216.}$$

The chief advantage of this method lies in the fact that any "give" in the supporting beam is eliminated by use of the second wire. Temperature changes are also practically eliminated. On the other hand, this method involves a rather accurate knowledge of "g," and requires an exceedingly accurate value of d.

Example. If, in a particular case, the wire under examination has a length of 3 meters and a diameter of approximately 1 millimeter, find what error in d will just compensate an error of 1 millimeter in l.

The method just described has been very much improved by Mr. G. F. C. Searle of the Cavendish Laboratory. He substitutes a sensitive level for the scale and vernier. The wires supported from a beam, as before, carry two brass circular frames FF' (Fig. 103). These frames are tied together by a brass strip C.

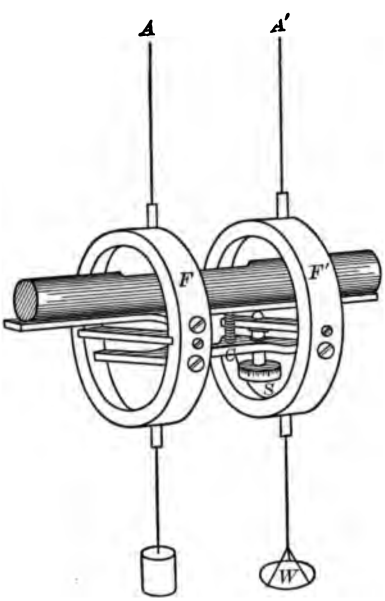


Fig. 103.

The level case is pivoted to one frame F, while the other end of the level rests upon a micrometer screw S. When the wire A' is stressed longitudinally by putting a load in the pan W, the corresponding elongation is measured on the micrometer screw with a high degree of accuracy.

(ii) *Case of a Thick Bar.* When the piece of metal to be examined is relatively thick, it is impossible by either of the preceding methods to secure an elongation sufficiently large to be measured accurately. Determinations of this kind are, therefore, generally made in the testing room of the Engineering Laboratory, by use of an "extensometer" which measures the elongation after it has been magnified in a known ratio. For this work, see Ewing's *Strength of*

Material, pages 73–83. Young's Modulus for a bar may also be determined by a method which involves the bending instead of the stretching of the specimen.

The Rigidity Modulus.

227. (i) *Case of a Wire.* The most convenient method for the determination of the shearing modulus, in this case, is to fasten one end of the wire to a beam in such a way that it cannot twist where clamped. To the lower (free) end of the wire attach a metal disk or bar of such a shape that its moment of inertia can be easily and accurately computed. Set the disk into horizontal vibration about the wire as an axis. Then we have (Eq. 58½, § 103), for the period of vibration of the disk,

$$T = 2\pi\sqrt{\frac{I}{K}},$$

where I is the moment of inertia and K the moment of force required to twist the disk through an angle of one radian. In order to determine n it remains only to determine K in terms of n. Let l be the length and r the radius of the wire. Consider the lowest cross-section of the wire, i.e., the one at the point where the disk is clamped on. In this cross-section, consider any point distant x from the axis of the wire. When the disk is twisted through an angle θ, this point will be displaced from its position of equilibrium by an amount $x\theta$.

Each cross-section throughout the entire length, l, of the wire will be sheared; and will be sheared by the same amount. The amount of this shear (i.e., shearing strain) will be $\frac{x\theta}{l}$. From this we can at once compute the shearing stress by multiplying the strain by n. Hence,

$$\left.\begin{array}{l}\text{Shearing stress, at point}\\ \text{distant } x \text{ from axis}\end{array}\right\} = \frac{nx\theta}{l} = \text{force on unit area.}$$

Consider now an elementary ring in this cross-section, having the axis of the wire for its center, x for its radius, and dx for its width.

The area of this ring is $= 2\,\pi x\,dx$.

Force exerted over area of ring $= \frac{nx\theta}{l} \cdot 2\pi x dx$.

$$\text{Moment of force over area of ring} = \frac{nx\theta}{l} \cdot 2\pi\,xdx \cdot x$$

$$= \frac{2\,\pi n\theta}{l} \cdot x^3\,dx.$$

To obtain the moment of force required to shear the entire cross-section to the amount it is sheared by twisting the disk through an angle θ, we have merely to add together the moments of force over each such elementary ring; thus, calling this total moment L, we have

$$L = \frac{2\,\pi n\theta}{l}\int_0^r x^3\,dx = \frac{\pi n\theta\,r^4}{2\,l}\,. \qquad \text{Eq. 217.}$$

But, by definition,

$$K = \frac{L}{\theta} = \frac{\pi n r^4}{2\,l},$$

and hence

$$T = 2\,\pi\sqrt{\frac{I}{\frac{\pi n r^4}{2\,l}}}, \qquad \text{Eq. 218.}$$

or

$$n = \frac{8\,Il\pi}{T^2 r^4}, \qquad \text{Eq. 219.}$$

which is the laboratory equation for the coefficient of rigidity.

A slight variation of this method in which the moment of inertia need not be explicitly computed is that in which "Maxwell's needle" is employed. The following design and description is that employed by Professor J. A. Ewing when at Cambridge.

The idea of Maxwell's needle is simply that of a torsional pendulum whose moment of inertia can be changed by an accurately measurable amount. It consists of a brass tube whose length we may call 2 *a*. Into this tube telescope two *solid* brass cylinders each of whose lengths is ½ *a*, and also two *hollow* brass cylinders, each of whose lengths is ½ *a*. These may be placed in either of the two positions shown in Fig. 104. Call the mass of each of the solid cylinders M_1, that of each of the hollow cylinders M_2. Then the change indicated between the two positions shown in Fig. 104 is equivalent to shifting two heavy particles of mass $M_1 - M_2$ from a distance ¾ *a* to a distance ¼ *a* from the axis of rotation. The change in moment of inertia thus produced is

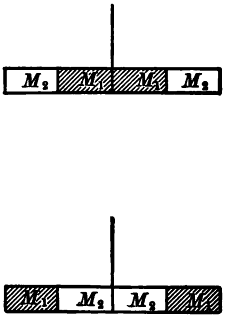


Fig. 104.

$$I_1 - I_2 = 2\,(M_1 - M_2)\,(\tfrac{9}{16}\,a^2 - \tfrac{1}{16}\,a^2) = (M_1 - M_2)\,a^2.$$

Let T_1 and T_2 be the periods of oscillation in the two cases respectively, then

$$\frac{T_1^2}{T_2^2} = \frac{I_1}{I_2},$$

and hence,

$$\frac{T_1^2}{T_1^2 - T_2^2} = \frac{I_1}{I_1 - I_2} = \frac{I_1}{(M_1 - M_2)\,a^2},$$

or

$$\frac{I_1}{T_1^2} = \frac{(M_1 - M_2)\,a^2}{T_1^2 - T_2^2}. \qquad \text{Eq. 220.}$$

But, Eq. 219,

$$n = \frac{8\,\pi\,l}{r^4}\cdot\frac{I_1}{T_1^2}\,.$$

Eliminating $\frac{I_1}{T_1^2}$ between 219 and 220, one obtains an equation for n, involving only quantities which are directly observable with a balance, a measuring stick, and a watch, the three fundamental instruments of the laboratory, namely,

$$n = \frac{8\pi l}{r^4} \cdot \frac{(M_1 - M_2)a^2}{T_1^2 - T_2^2}. \qquad \text{Eq. 221.}$$

Using a fixed length of wire, and the same known masses, one has only to measure a single radius and two periods in order to determine the rigidity of any sample of wire. This " needle " is described by Maxwell in his Bakerian Lecture on the " Viscosity of Gases, " *Phil. Trans.*, Vol. 156, 1866.

228. (ii) *Case of a Thick Bar.* Use torsion lathe in Engineering Laboratory.

CHAPTER VI.

FLUID MOTION.

229. This study of fluids comprises two chapters, the first of which deals with fluids at rest, the second with fluids in motion. The entire subject of fluid motion is frequently discussed under the head of Hydrodynamics, while these two subdivisions are called Hydrostatics and Hydrokinetics. Hydromechanics is sometimes used to describe the entire subject, while Hydraulics, which has to do properly with the motion of fluids in pipes and canals, is often employed as a synonym for Hydrokinetics.

HYDROSTATICS.

230. The fundamental property of fluids, which is used as a basis of Hydrostatics, is the fact, implied in the definition of a fluid (§ 215), that *when a fluid is in equilibrium the stress across any surface drawn* in it has no tangential component; in other words, the stress is entirely normal to the surface.

When the various parts of any *actual* fluid are *in relative motion,* the fluid always exhibits tangential stresses. If, for instance, in water this were not so, we should not be able, by twirling a bucket of water about a vertical axis to set the water into rotation. This property of liquids in motion — in virtue of which they offer resistance to change of shape — is called *viscosity,* or sometimes *internal friction.* It should be carefully distinguished from the resistance which solids offer to change of shape; for in solids the resistance is proportional to the *amount* of distorsion, while in fluids the resistance is proportional to the *rate* of distorsion. In Hydrostatics we shall deal only with fluids in *equilibrium*, and therefore viscosity does not here enter into consideration.

Pressure at any Point in a Fluid the Same in all Directions.

231. From the definition of a fluid body as one which is unable to sustain a shearing stress, it follows that at any point in the fluid the pressure — force per unit area — is the same in all directions. The physical meaning of this statement may be a little clearer when the theorem is derived *de novo* in the following classical manner, even though this property has already been derived, from another point of view, in the chapter on Elasticity, § 215.

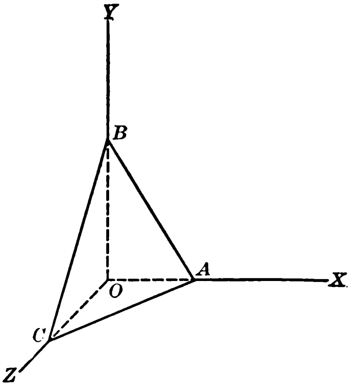


Fig. 105.

Let O be any point in the fluid, and with O as origin construct a system of rectangular axes OX, OY, and OZ, Fig. 105. Let any plane whose direction cosines are l, m, n cut these axes in the points A, B, C, and thus form a small tetrahedron $ABCO$. Now since, by hypothesis, this tetrahedron is in equilibrium, it must follow from the general equations of motion (107) that the sum of all the forces acting upon the tetrahedron along any direction are exactly equal to zero. Since the axes in Fig. 105 are chosen at random we may take the direction of the X-axis as typical of all directions. The only forces urging the particle — for we may so call the tetrahedron — along the X-axis are the forces acting upon the faces *BOC* and *BAC*. Let pressures along the three axes respectively be denoted by p_x, p_y, and p_z; while the pressure over the face *BAC* is called p and the area of the face *BAC* is called Δ.

Force over face *BOC*, parallel to X-axis $= + p_x \cdot l\Delta$.
Force over face *BAC*, normal to this face $= p \cdot \Delta$.
Component of force over face *BAC*, parallel to X-axis

$$= - pl \cdot \Delta.$$

Sum of all forces on particle parallel to X-axis

$$= p_x l \, \Delta - pl \cdot \Delta = 0.$$

Hence $p_x = p$.

The plain meaning of this is that the pressure over any element of area *BAC*, chosen at random, is equal to the pressure in the direction of X-axis, which is also chosen at random. We may therefore write for any fluid in equilibrium

$$p_x = p_y = p_z = p. \qquad \text{Eq. 222.}$$

As every student knows from experience, neither gases nor liquids are capable of sustaining any considerable tension — one might almost say any appreciable tension. All stresses in fluids will, therefore, in the following pages, be considered as pressures.

Equations of Equilibrium for a Fluid at Rest under any Forces.

232. While it has just been shown that, at any one point in a fluid, the pressure is constant in all directions, it is a matter of every-day experience that the pressure in any fluid varies, in general, from one point to another.

Imagine a parallelopiped of infinitesimal dimensions drawn at any point in the fluid with its edges parallel to the rectangular axes of coördinates X, Y, Z. Let α, β, γ denote the lengths of the edges of this parallelopiped and ρ the density of the fluid. Then $\rho\alpha\beta\gamma$ is the mass of the particle of fluid whose equations of equilibrium are sought. These equations are obtained as usual, by equating to zero all the forces acting upon the element of the fluid. These forces fall into two groups (1) the pressure, i.e., those forces which act over certain surfaces, and (2) the external or " body " forces, i.e., those which act upon each particle of the element. The typical force of the second kind — and almost the only one of this class which is met in practice — is that of gravity. The forces which act in a direction parallel to the axis of X are the pressure p over the left-hand YZ-face of the parallelopiped, the pressure p' over the right-hand YZ-face, and the external force whose acceleration (force on unit mass) we may call X. Equating these three forces to zero, we obtain the condition of equilibrium, namely,

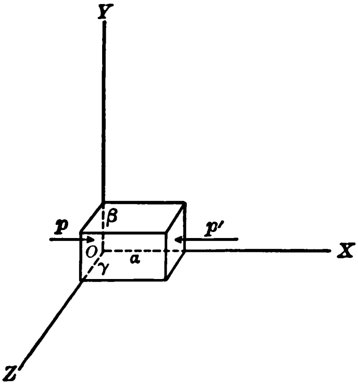


Fig. 106.

$$p\beta\gamma - p'\beta\gamma - \rho\alpha\beta\gamma X = 0. \qquad \text{Eq. 223.}$$

But by Taylor's theorem,

$$p' = p + \frac{\partial p}{\partial x}\alpha.$$

Introducing this value into Eq. 223 and dividing through by $\alpha\beta\gamma$ we obtain the first of the following three equations: the last two are gotten by an identical process,

$$\left.\begin{aligned} \rho X - \frac{\partial p}{\partial x} &= 0, \\ \rho Y - \frac{\partial p}{\partial y} &= 0, \\ \rho Z - \frac{\partial p}{\partial z} &= 0. \end{aligned}\right\} \qquad \text{Eq. 224.}$$

The physical meaning of these equations is that the space variation of the pressure in any direction is equal to the resolved force per unit of volume in that direction.

In any particular problem it will usually be required to find the pressure and hence to integrate these general equations. The following paragraph will serve as an illustration.

Case of a Fluid at Rest under Gravity.

233. Let us suppose that the only external force acting upon the fluid is its own weight; then with the usual choice of axes, in which that of Y is vertical, the conditions of the problem are expressed as follows:

$$X = Z = 0 \text{ and } Y = -g.$$

So that our general equations become

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial z} = 0 \text{ and } \frac{\partial p}{\partial y} = -\rho g \qquad \text{Eq. 225.}$$

Let us next assume what is practically true in the case of water and most liquids, namely, that the density of the fluid does not vary with the depth, y; this assumption is expressed by writing

$$\rho = constant;$$

then on integrating, Eq. 225, one has

$$p = -\rho g y + C,$$

where C is a constant of integration which may be determined by placing $y = 0$. Suppose the origin of coördinates to lie in the bounding surface of the fluid; then if the surface is free — and subject only to atmospheric pressure which we may denote by B — the boundary condition becomes $C = B$, and hence

$$p = -\rho g y + B. \qquad \text{Eq. 226.}$$

As the point at which p is measured advances downwards into the liquid, the value of y becomes negative and hence the pressure increases.

If the surface of the liquid in which the origin lies is not free, but is bounded by the walls of a vessel in which the liquid is subject to pressure — as for instance some part of a steam pump — then B assumes the value of this external pressure, whatever it may be.

If the atmospheric pressure be taken as a zero from which all other pressures are measured — as in the case of the Bourdon gauges ordinarily used on steam boilers — then $B = 0$, and the general equation becomes

$$p = -\rho g y. \qquad \text{Eq. 227.}$$

The physical meaning of this equation is that, under gravity, the pressure varies as the depth and is independent of the form of the containing vessel; a fact which is generally known as the " Hydrostatic Paradox."

VARIATION OF ATMOSPHERIC PRESSURE WITH ALTITUDE.

The Barometer.

234. Returning to the general equations (224), and retaining the same reference coördinates as employed in the preceding paragraph, one has

$$\partial p = -\rho g \partial y, \qquad \text{Eq. 228.}$$

an equation which is true in any fluid, gaseous or liquid, so long as no external forces except that of gravity are acting upon the fluid. But before integration of this equation is possible one must express ρ as a function either of y or of p. Now this is precisely what Boyle's Law does in the case of a gas.

Accordingly, the pressure in the earth's atmosphere at any given temperature t and at any assigned altitude y is completely determined by the general equation 228 and the condition

$$\frac{p}{\rho} = Rt^{\circ} = constant \quad \text{(Boyle's Law)},$$

in which R is the gas constant and has a value of 2.87×10^{6} for air, when p and ρ are measured on the C.G.S. system of units, and temperatures are referred to the absolute Centigrade scale. Eliminating ρ between these two equations, we obtain

$$\frac{\partial p}{p} = -\frac{g}{Rt^{\circ}}\partial y$$

which on integration yields

$$\log p = -\frac{g}{Rt^{\circ}}y + \log C,$$

and hence

$$p = C\varepsilon^{\frac{-gy}{Rt}}.$$

Let us denote by B_0 the value of the atmospheric pressure at sea level; then if the origin of coördinates also be chosen at sea level, we have

$$p = B_0 \, \varepsilon^{\frac{-gy}{Rt}}. \qquad \text{Eq. 229.}$$

The student will find it an interesting exercise to prove that if the barometer readings, reduced to dynes per square centimeter, at any two stations at any one instant are p_1, and p_2 respectively, the difference in altitude between the stations is

$$y_2 - y_1 = \frac{Rt^o}{g} \log_e \frac{p_1}{p_2} = \frac{Rt^o}{g} \times 2.3026 \log_{10} \frac{p_1}{p_2} \qquad \text{Eq. 230.}$$

It is evident that in practice a temperature correction would have to be applied to this expression.

CENTER OF PRESSURE.

235. When a liquid is at rest in a vessel, we have just seen that the pressure varies directly as the depth. Under these conditions it frequently becomes a matter of great importance to determine through just which point on any plane surface the resultant of all these various forces acts. This point, which we shall proceed to show is a perfectly definite one, is called the *center of pressure.*

Consider any point (x, y) on an immersed plane surface, the surface itself having been selected as the XY-plane. Denoting pressures by p, the force over any element of area $dx\, dy$ will be $p\, dx\, dy$ in amount, while its direction will be normal to the surface. The total force exerted on the entire plane S will, therefore, be

$$F = \int\int_0^S p \, dx \, dy.$$

To determine the point $(\bar{x}, \bar{y})$ where this resultant force, F, acts, we have then merely to find a point about which the sum of the moments of all these elementary forces $p\,dx\,dy$ is zero. Accordingly, the following equations serve both to define and to compute the center of pressure:

$$\bar{x} = \frac{\int\int_0^S xp\,dx\,dy}{\int\int_0^S p\,dx\,dy}, \qquad \bar{y} = \frac{\int\int_0^S yp\,dx\,dy}{\int\int_0^S p\,dx\,dy}. \qquad \text{Eq. 231.}$$

These equations, it will be observed, are identical in form with those for the center of mass of a plane surface (Eq. 60^{IV}, § 59). One may therefore regard the determination of the center of pressure for any particular plane surface as the determination of center of mass for a lamina whose surface density, σ, is variable, and has at every point the same value as the pressure at that point.

It will be well for the student here to prove that the center of pressure of a rectangle, having one edge in the free surface of the liquid, lies at a depth equal to ⅔ that of the rectangle. Does the center of pressure vary as the plane of the rectangle is inclined to the surface of the liquid?

CENTER OF BUOYANCY.

236. Since, as we have already learned, the pressure of any liquid upon the bottom of any floating vessel is proportional to the depth, and equal to $\rho g y$, it follows that the force over each element of area of the submerged surface is equal to the weight of a vertical column of liquid which will just cover this element of area and will reach to the free surface of the liquid. Accordingly, one may replace this buoyant force by the resultant of the weights of these various columns of water; i.e., by the weight of the fluid displaced. This is known as *Archimedes' Principle.* But for many purposes it is very important to know *through what point* of the floating body this buoyant force acts; i.e., to find the center of pressure. From what precedes, it is clear that having replaced pressure by weights, *the buoyant force will act vertically through the center of mass of the displaced fluid.* Hence the buoyant force is completely determined, its amount, direction, and point of application being known.

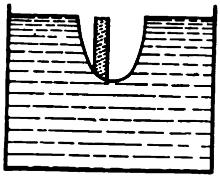


Fig. 107.

EQUILIBRIUM OF A FLOATING BODY.

237. The conditions which determine equilibrium in a floating body are the same as those which apply to other bodies; namely,

(i) Sum of all external forces must vanish.
(ii) Sum of all external torques must vanish.

The first of these conditions is satisfied when the body sinks in the fluid to such a depth that the weight of the fluid displaced is equal to the weight of the vessel. The second condition is satisfied when the center of buoyancy lies in the same vertical line with the center of mass of the vessel.

CRITERION OF STABILITY IN A FLOATING BODY. THE METACENTER.

238. When equilibrium has once been obtained, the question which arises is as to whether it is of stable, neutral, or unstable, type; in other words, what will happen to the vessel if it is slightly rotated? Let *B*, Fig. 108, denote the center of buoyancy, and *G* the center of gravity of the floating vessel; then if the vessel be slightly rotated, the cross-section of the displaced fluid will be altered, and, in general, the center of buoyancy will be shifted as shown in Fig. 109. Let us call the new center of buoyancy *B′*. Then the righting force of the fluid pressure will act along the vertical line *B′M*. *The point M where the vertical through the center of buoyancy in any position of the vessel intersects the vertical through the center of buoyancy in the position of equilibrium is called the metacenter.*

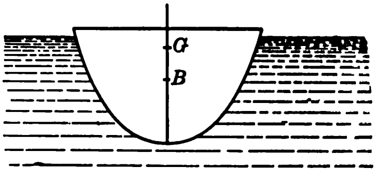


Fig. 108.

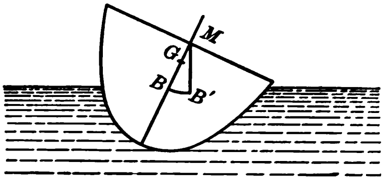


Fig. 109.

Another way of looking at the metacenter is to consider it as the center of curvature of the arc described by the center of buoyancy as the vessel rolls or pitches in the fluid.

So soon as the positions of *M* and *G* are known for any particular body, the kind of stability is at once determined; for evidently, if *M* lies above *G*, the torque will be such as to right the vessel; but, if *M* lies below *G*, the torque will be such as to increase the angular displacement of the vessel and upset it.

The angular stability of a ship when displaced through any given angle is then merely a question of the distance from its center of gravity to its metacenter, — *GM* in Fig. 109, — a quantity known as the metacentric height. The student who desires to look into this interesting question more fully, should consult Sir William White's *Manual of Naval Architecture*, Chap. III; and also the article on "*Hydrostatics*," *Ency. Brit.*, where the following simple expression for the distance between metacenter and the center of buoyancy BM (Fig. 109) is derived,

$$BM = \frac{\text{Moment of inertia of water-line area}}{\text{volume of displacement}},$$

an expression which allows one to compute immediately the torque which tends to right a vessel displaced at any given angle.

HYDROKINETICS.

239. When a particle of fluid is in equilibrium, we have seen that this state of affairs is described in precisely the same manner as for solid bodies; namely, by equating to zero the sum of all the external forces acting upon the fluid particle. Grouping these forces under two heads, body forces and variations of pressure, we obtained, § 232, for the equilibrium of unit mass,

$$\left(X - \frac{1}{\rho}\frac{\partial p}{\partial x}\right) = 0, \qquad \text{Eq. 224.}$$

and two similar equations. But let us consider now the general case in which the liquid is not in equilibrium. Then the only change we shall have to make will be to equate the external forces to the mass acceleration. In other words, we shall have to modify Eq. 224 in accordance with Newton's Second Law of Motion.

First, let us limit our discussion to an element of the fluid so small that we may treat it as a particle; that is, let us choose a volume of fluid so minute that we may neglect any difference in velocity between the various parts of the element. We may select a parallelopiped whose sides are dx, dy, dz, respectively; then the mass of the element will be $\rho\,dx\,dy\,dz$. If u, v, w be components of the velocity of this element parallel to the axes of X, Y, Z, respectively, the momentum along the axis of X will be given by $\rho\,dx\,dy\,dz\,u$, and the mass acceleration along the X-axis will be $\rho\,dx\,dy\,dz\,\frac{du}{dt}$ and two similar expressions for the other two axes. This is the expression which must be equated to the sum of all the external forces acting upon the particle in order to give us the equation of motion. Hence we have

$$\rho\,dx\,dy\,dz\frac{du}{dt}=\rho\,dx\,dy\,dz\left(X-\frac{1}{\rho}\frac{\partial p}{\partial x}\right)$$

and two similar equations.

It is to be insisted upon here that the differential coefficient $\frac{du}{dt}$ means the *complete* differentiation of u with respect to t, and indicates the acceleration along the X-axis due to all causes whatsoever. Dividing through by the mass of the particle, we obtain as the equations of motion for unit mass of the fluid,

$$\left.\begin{aligned}\frac{du}{dt}&=X-\frac{1}{\rho}\frac{\partial p}{\partial x},\\ \frac{dv}{dt}&=Y-\frac{1}{\rho}\frac{\partial p}{\partial y},\\ \frac{dw}{dt}&=Z-\frac{1}{\rho}\frac{\partial p}{\partial z}.\end{aligned}\right\}\qquad\text{Eq. 232.}$$

It remains only to put the left-hand member into a more workable form. To do this we have first to understand that the velocity of a particle in any given fluid depends only upon two things: namely, upon the position of the particle, xyz, and upon the time, t. Thus if one assumes a fixed position at any point in a pipe conveying water, the velocity of the water at that point may change from instant to instant according as a tap is opened or closed at some other point in the pipe. The velocity of the water at the fixed point might be described as a function of the time alone. But consider now a particle of water moving from point to point; its velocity is liable to change not only because of a tap that may have been opened, but also because the particle is moving into a new region of the pipe, where the velocity is different from that at which the particle started. Thus, even if the motion is absolutely uniform from instant to instant, even if the motion be perfectly steady, and there be a constriction in the pipe, a particle will increase in velocity as it passes from the larger to the smaller cross-section.

Accordingly, it is clear that each component of the velocity will, in general, be a function of four variables, x, y, z, and t. This being so, we have by the rules of ordinary differentiation, using round deltas to indicate partial differentials,

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}\\ &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},\end{aligned}$$

and two similar equations for $\frac{dv}{dt}$ and $\frac{dw}{dt}$.

The physical meaning of the first term on the right is the rate at which u varies *with the time* at any one *fixed* point xyz; while the last three terms in the right-hand member tell us how the velocity of a particle is varying at any one instant, t, *owing to its change of position*. Substituting this value of $\frac{du}{dt}$ in Eq. 232, one gets the equations of motion, as first derived by Euler, in their final form, namely,

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= X - \frac{1}{\rho}\frac{\partial p}{dx}, \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= Y - \frac{1}{\rho}\frac{\partial p}{dy}, \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= Z - \frac{1}{\rho}\frac{\partial p}{dz}. \end{aligned}\right\} \qquad \text{Eq. 233.}$$

240. The problem, in general, will be to find the five unknown quantities which appear in these three general equations; namely, u, v, w, p, and ρ.

It is clear, then, that we shall need two more equations, which are obtained as follows: first, there will always be given some description of the fluid which will enable us to write a mathematical relation between ρ and p. Thus, if the fluid be a gas changing volume at a constant temperature, we may describe this fact by writing

$$p = \rho R t^{\circ},$$

or

$$\frac{p}{\rho} = \textit{constant}.$$

If the fluid be an incompressible liquid — as is practically the case with water — this physical property would be expressed by writing

$$\rho = \textit{constant}. \qquad \text{Eq. 234.}$$

The fifth and last equation is obtained by putting into algebraic form the law of the conservation of matter. Let us consider any element of volume ∂x, ∂y, ∂z. The amount of matter contained in this volume at any instant t is $\rho\, \partial x\, \partial y\, \partial z$. And the rate at which the matter in this volume is changing is $\partial x\ \partial y\ \partial z\ \frac{\partial \rho}{\partial t}$. *But the rate at which matter is changing in any given volume is simply the rate at which matter is entering that volume minus the rate at which matter is leaving that volume.* This difference we now proceed to compute.

Let us suppose that the center of the element is located at the point (xyz). Then the amount of matter entering the left-hand YZ-face of the element, during the interval ∂t, is

$$\rho u\ \partial y\ \partial z\ \partial t.$$

That which *leaves* through the right-hand face will be, say, $\rho' u'\ \partial y\ \partial z\ \partial t$ where, by Taylor's Theorem,

$$\rho' u' = \rho u + \frac{\partial(\rho u)}{\partial x}\, dx,$$

whence $\rho u\ \partial y\ \partial z\ \partial t - \rho' u'\ \partial y\ \partial z\ \partial t = -\ \partial x\ \partial y\ \partial z\ \frac{\partial(\rho u)}{\partial x}\ \partial t.$

Similar expressions may be written for the upper and lower, as well as for the front and back, faces of the element, so that we have for the total increase of matter in the element during the time dt, the sum of three such expressions, namely,

$$\partial x\ \partial y\ \partial z \frac{\partial \rho}{\partial t}\ \partial t = -\ \partial x\ \partial y\ \partial z\ \partial t \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right],$$

from which we may write

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0, \qquad \text{Eq. 235.}$$

as the final form of this expression, which is known as " the equation of continuity."

241. Having now obtained five general equations between five unknown quantities, we proceed to a few simple applications, which will serve to illustrate their meaning and use.

Special Cases of Fluid Motion.

242. Case I. *Very Slow Motions.* When the squares of the velocity and its first differential coefficients with respect to x, y, and z, may be neglected, the general equations (233) become

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= X - \frac{1}{\rho}\frac{\partial p}{dx}, \\ \frac{\partial v}{\partial t} &= Y - \frac{1}{\rho}\frac{\partial p}{dy}, \\ \frac{\partial w}{\partial t} &= Z - \frac{1}{\rho}\frac{\partial p}{dz}. \end{aligned}\right\} \qquad \text{Eq. 236.}$$

243. Case II. *Steady Motion.* When at *any given point* in a fluid the motion does not change from one instant to another, the motion is said to be *steady.* This condition is expressed algebraically as follows:

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = 0, \qquad \text{Eq. 237.}$$

and hence the general equations become

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= X - \frac{1}{\rho}\frac{\partial p}{dx}, \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= Y - \frac{1}{\rho}\frac{\partial p}{dy}, \\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= Z - \frac{1}{\rho}\frac{\partial p}{dy}. \end{aligned}\right\} \text{Eq. 238.}$$

In any case of steady motion, the paths which are followed by the various particles of fluid are called *Stream Lines.* Let q denote the total velocity of the particle; then q will be defined by the following equation:

$$q^2 = u^2 + v^2 + w^2.$$

Also let *ds* denote an element of the stream line of the particle under consideration. Then,

$$\left.\begin{aligned} u &= q\,\frac{\partial x}{\partial s}, \\ v &= q\,\frac{\partial y}{\partial s}, \\ w &= q\,\frac{\partial z}{\partial s}, \end{aligned}\right\} \qquad \text{Eq. 239.}$$

and substituting these values in the general equations for steady motion, we have

$$\left.\begin{aligned} q\frac{\partial x}{\partial s}\frac{\partial u}{\partial x} + q\frac{\partial y}{\partial s}\frac{\partial u}{\partial y} + q\frac{\partial z}{\partial s}\frac{\partial u}{\partial z} &= q\frac{du}{ds} = X - \frac{1}{\rho}\frac{dp}{dx}, \\ q\frac{\partial x}{\partial s}\frac{\partial v}{\partial x} + q\frac{\partial y}{\partial s}\frac{\partial v}{\partial y} + q\frac{\partial z}{\partial s}\frac{\partial v}{\partial z} &= q\frac{dv}{ds} = Y - \frac{1}{\rho}\frac{dp}{dy}, \\ q\frac{\partial x}{\partial s}\frac{\partial w}{\partial x} + q\frac{\partial y}{\partial s}\frac{\partial w}{\partial y} + q\frac{\partial z}{\partial s}\frac{\partial w}{\partial z} &= q\frac{dw}{ds} = Z - \frac{1}{\rho}\frac{dp}{dz}. \end{aligned}\right\} \qquad \text{Eq. 240.}$$

BERNOUILLI'S THEOREM.

244. Let us now assume, what is generally the case, that the only body force acting upon the fluid is that of gravity. This condition is $X = Z = 0$, and $Y = -g =$ *constant.* Then, if we make another assumption which is also practically true in nearly every case, — namely, that the fluid is incompressible; that is, $\rho =$ *constant,* — we find Eq. 240 easily integrable, for if we multiply each of the last two members by $\frac{dx}{ds}$, $\frac{dy}{ds}$, and $\frac{dz}{ds}$ (which are merely direction cosines of *s*) respectively, and then add, we obtain

$$u\frac{du}{ds} + v\frac{dv}{ds} + w\frac{dw}{ds} = -g\frac{dy}{ds} - \frac{1}{\rho}\frac{dp}{ds}. \qquad \text{Eq. 241.}$$

If now this expression be integrated with respect to s, — that is, if it be integrated along a stream line, — we obtain

$$\tfrac{1}{2}(u^2 + v^2 + w^2) = -gy - \frac{p}{\rho} + constant.$$

Let us denote this "*constant*" by H, and remember that H is a constant along any particular stream line, but may vary from one stream line to another. On transforming, one obtains

$$\tfrac{1}{2}q^2 + gy + \frac{p}{\rho} = H, \qquad \text{Eq. 242.}$$

which is Bernouilli's Theorem.

The physical meaning of this theorem is that the energy which is available *in each unit of mass* in a fluid is made up of three elements: (i) the kinetic energy $\frac{1}{2}q^2$, (ii) the gravitational energy gy, and (iii) the energy which comes from a pressure exerted through a volume, $\frac{p}{\rho}$. For since ρ is the density, $\frac{1}{\rho}$ is the specific volume of the fluid; hence $\frac{p}{\rho} = p \times$ *specific volume.* This third term represents the energy required to pump, say, a gram of water, into the bottom of a tank, after the gram has been lifted to the same level as the bottom of the tank.

The advanced student will find that Bernouilli's Theorem plays a large and important rôle in the subject of hydraulics. This theorem may be derived from energy considerations alone, as is done, for instance, in Bovey's *Hydraulics*, pages 8–10 (ed. 1901). Every student of this subject should read W. Froude's classical address on "The Theory of Stream Lines in Relation to the Resistance of Ships," *Nature,* 13, page 50 (1875).

245. Case III. *Velocity of Efflux of a Liquid under Gravity. Torricelli's Theorem.*

Let a tank of liquid be provided with an orifice in one side as shown in Fig. 110, and let the origin of coördinates be chosen in the *upper* surface of the liquid. We may consider the area of the upper surface so large compared with that of the orifice that the velocity of the upper surface can be neglected. With these conventions, it is obvious that when y vanishes, q will vanish, and the pressure will equal the atmospheric pressure, which latter we may call B. Making these substitutions in Bernouilli's Theorem, Eq. 242, we have

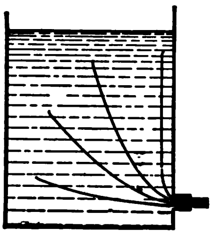


Fig. 110.

$$\frac{B}{\rho} = H,$$

an equation which determines the constant H; so that our general equation becomes

$$\tfrac{1}{2} q^2 + gy + \frac{p}{\rho} = \frac{B}{\rho}. \qquad \text{Eq. 243.}$$

At the surface of the jet which leaves the orifice we also have the condition $p = B$. Introducing this condition into the general equation, we have, at the orifice,

$$\tfrac{1}{2} q^2 + gy = 0,$$

or

$$q = \sqrt{-2\,gy}, \qquad \text{Eq. 244.}$$

which is Torricelli's Theorem — evidently a special case of Bernouilli's. The interesting corrections which must be applied to this theorem before it can be safely used to compute the outflow of liquids, and the phenomena of the *vena contracta*, must be reserved for advanced study.

246. Case IV. *A vessel of liquid rotating with constant angular velocity about a vertical axis under no forces except gravity. Problem. Find the form of the free surface.*

Let ω denote the constant angular velocity, and let y be selected for the vertical axis of coördinates. Then the conditions of the problem are completely described by the following equations:

$$\left.\begin{array}{l} X = 0, \\ Y = -\,g, \\ Z = 0, \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l} u = -\,\omega z, \\ v = 0, \\ w = +\,\omega x. \end{array}\right\}$$

Accordingly, the general equations of motion (233) become

$$\left.\begin{array}{r} -\,\omega^2 x = -\,\dfrac{1}{\rho}\,\dfrac{\partial p}{dx}\,, \\ 0 = -\,g - \dfrac{1}{\rho}\,\dfrac{\partial p}{dy}\,, \\ -\,\omega^2 z = -\,\dfrac{1}{\rho}\,\dfrac{\partial p}{dz}\,. \end{array}\right\} \qquad \text{Eq. 245.}$$

Now it will be observed that each of these "partial differential equations" may be obtained by differentiating the following expression, which is therefore *an* integral of Eq. 245:

$$p = \tfrac{1}{2}\,\rho\omega^2\,(x^2 + z^2) - \rho g y + \textit{constant}. \qquad \text{Eq. 246.}$$

To find the form of the free surface, we have now merely to place $p = \textit{constant}$, say B. This constant B may also include the constant of integration; and then since the motion does not vary with time, we have for the equation of the free surface,

$$x^2 + z^2 - 2\,\frac{gy}{\omega^2} = \frac{2\,B}{\omega^2\rho} = \textit{constant}, \qquad \text{Eq. } 247.$$

which is evidently a paraboloid of revolution, concave on the upper side, and with a latus rectum $= \frac{2\,g}{\omega^2}$. This principle has been applied in one type of speed gauge widely used in automobiles.

For a more advanced consideration of these subjects and of the beautiful problem of vortex motion, the student is referred to Webster's *Dynamics of Particles, and of Rigid, Elastic, and Fluid Bodies*, and, of course, to Lamb's already classical treatise, *Hydrodynamics* (Cambridge, 1895).

INDEX

The Numbers Refer to Pages